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DEVOTED TO THE INTERESTS OF MATHEMATICS IN JUNIOR AND SENIOR HIGH SCHOOLS

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THE MATHEMATICS TEACHER

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CALCULUS IN THE HIGH SCHOOL[1]

By J. M. KINNEY
Crane Junior College, Chicago

Social value of mathematics. Primitive man began to think quantitatively as soon as he made a conscious attempt to adjust means to an end. Thus the Egyptians developed quite a bit of mathematical lore before historic times. They, together with other prehistoric peoples, employed their mathematical knowledge in surveying land, building temples, carrying on commerce, in navigation, astronomy, and mechanics. From prehistoric times to the present mathematics has been employed increasingly in the solution of problems in widely different fields. Today is the golden age not only of theoretical, but of applied mathematics. It is being employed in all fields of scientific investigation. It is employed not only in physics, mechanics, and chemistry, but in such fields as biology, psychology, medicine, geology, economics, pedagogy, sociology, and business. In fact it is the ideal of every science to formulate from the mass of data it has collected, laws which may be expressed by mathematical formulas. Let me cite an example of and a quotation on the use of mathematics in fields which most of us have felt to be nonmathematical.

During the World War Dr. Alexis Carrel collected data bearing on the size of wounds and the time required for healing. This data was plotted and the equation of the curve was determined. It turned out to be an exponential equation by means of which any abnormality in the healing of the wound could be detected even before there was any visible indication of infection.

Professor Mayo Smith in his book on Statistics and Sociology says, "Statistics is one extremely useful instrument of investigation in the study of sociology. It has lately been affirmed that it is the most useful and potent instrument that science possesses, and there has been and is a tendency among some

[1] This paper was read at a meeting of the Chicago Mathematics Club, February 16, 1923.

writers to expand statistics, so that it becomes in itself a scheme of social science, covering all the phenomena of social life."

The most valuable branch of mathematics that is used in making investigations in scientific and technical fields is the calculus. It is therefore important that students working in these fields should master its technique as early in their careers as they are able to comprehend its fundamental principles.

The fundamental notions of the calculus. There are the three fundamental notions of the calculus. They are called the derivative, the anti-derivative, and the definite integral.[1]

Let us examine these notions briefly for the purpose of making an estimate of the amount of preparation required of a student to understand them. Consider the derivative. For illustration of the principles involved let us find the instantaneous speed of a ball thrown vertically upward with an initial velocity of 80 feet per second after 2 seconds. The formula giving the height, h feet, after t seconds is

$$h = 80t - 16t^2.$$

Consider an interval of time, $\triangle t$, which we shall look upon as variable and extending from $t = 2$ to $t = 2 + \triangle t$. For these respective values of t we have

$$h = 80 \cdot 2 - 16.2^2$$

$$h = 80(2 + \triangle t) - 16(2 + \triangle t)^2$$

$$= 80 \cdot 2 + 80\triangle t - 16.2^2 - 64\triangle t - 16\triangle t^2$$

The difference of these heights is

$$h = 16 \triangle t + 16 \triangle t^2.$$

To find the average velocity during the interval we divided by $\triangle t$ and get

$$\frac{\triangle h}{\triangle t} = 16 - 16\triangle t.$$

Now allow $\triangle t$ to become smaller and smaller. Evidently the average velocity becomes more nearly equal to the actual or instantaneous velocity at the time $t = 2$ seconds. And it becomes exactly equal to the instantaneous velocity of $t = 0$.

We have illustrated here a variable interval of time approaching a limit 0, and a variable average velocity approaching a limit 16.

[1] For an interesting discussion of these notions see the chapter in Monographs on Topics of Modern Mathematics, entitled "The Function Concept and the Calculus," by Professor G. A. Bliss. Edited by Professor J. W. A. Young. Also see Introduction to "Mathematical Analysis" by Professor F. L. Griffin, Houghton Mifflin Co.

Again, let us compute the direction of the tangent to the curve $y = x^2$ at the point $P = (3, 9)$. This tangent may be defined as the limit approached by a secant passing through P and another point on the curve, $Q = (3 + \triangle x, 9 + \triangle \triangle y)$. The slope of this secant is $\frac{\triangle y}{\triangle x}$. Since $9 + \triangle y = (3 + \triangle x)^2$, we find $\triangle y = 6 \triangle x + \triangle x^2$. Hence $\frac{\triangle y}{\triangle x} = 6 + \triangle x$. Now as Q approaches $P \triangle x$ approaches 0. Hence the limit approached by $6 + \triangle x$ is 6 and this is the slope of the tangent. Thus the variable slope of the moving secant has for its limit the fixed slop of the tangent.

The notion of a limit approached by a variable as exhibited in the two illustrations above should have been brought to the attention of the student during his study of plane geometry in connection with the measure of the angle formed by a secant and a tangent. Thus let Q, R, S, and T be four points on a circle. Let the secants QR and ST meet in P. Now think of R, Q, and S as being fixed points and let T approach S. For all positions of T the value of $\angle P$ is given by the formula $\angle P = \frac{1}{2}(\overset{\frown}{RT} - \overset{\frown}{SQ})$. If T is allowed to coincide with S the same formula holds and the pupil sees that the case of the measure of an angle formed by a secant and a tangent is one in which the angle may be considered as being the limit approached by the angle formed by two secants.

The two illustrations above with numerous others may be used to develop the notion in the mind of the student that the derivative of a function, $y = f(x)$, with respect to x is the instantaneous rate at which y is changing per unit change in x.

Let us now reverse the process. That is to say, having given the rate at which a quantity is increasing to find the magnitude of the quantity. By way of illustration let $y = 3x^2$. Let us think of y as being the rate at which a function, $\delta(x)$, is increasing. That is to say y is the derivative of $\delta(x)$. To find this function we need to find a function which differentiated will give $3x^2$. By inspection we see that it is $x^3 + c$. Thus the antiderivative or integral of $3x^2$ with respect to x is $x^3 + c$.

In this connection let us consider a remarkable relation existing between the rate of change of the area under the curve,

$y = f(x)$ bounded by two ordinates KL (fixed) and PQ (varying) and the ordinate $PQ = y$. Let us denote this area by A. Let $OP = x$. Assuming that $y = f(x)$ is a continuous function, that is, there are no breaks in the curve, it follows that A is a function of x, let us say $A = \delta(x)$. Now, let us give an increment, $PR = \triangle x$, to x, then we have a corresponding increment $PQSR = \triangle A$ given to A. Evidently the area $PQSR$ is equal to the area of a rectangle with base $PR = \triangle x$ and an altitude equal to an ordinate $\overline{y}$ lying between ordinates $PQ = y$ and $RS = y + \triangle y$. That is

$$\triangle A = \overline{y} \triangle x$$

$$\text{or } \frac{\triangle A}{\triangle x} = \overline{y}.$$

Let $\triangle x \to 0$ then $\overline{y} \to y$. Hence

$$\frac{dA}{dx} = y.$$

That is to say the rate of change of the area $A = \delta(x)$ is equal to the length of the varying ordinate $PQ = y$.

It follows, therefore, that

$$A = \int y dx$$

Let us apply this formula to the finding of the area under the curve $y = 3x^2$ bounded by the ordinates whose abscissas are $OK = 3$ and $OP = 5$. Substituting $3x^2$ for y in the formula we have

$$A = \int 3x^2 dx$$

$$\text{or } A = x^3 + C$$

Since the varying area starts from KL, $A = 0$ if $OK = 3$. Substituting this pair of values of A and x in the formula we obtain

$$0 = 3^3 + C$$

$$\text{or } C = 27.$$

We now have for other values of x

$$A = x^3 - 27.$$

If $x = 5$ then $A = 125 - 27 = 98$.

Let us recall that $A = \delta(x) = \int y dx$. If $OK = a$ and $OP = b$ it follows that the area under the curve $y = f(x)$ is equal to $\delta(b) - \delta(a)$. That is to say we can find the integral under any continuous curve provided it is possible to find the integral $\delta(x)$ of $f(x)$.

The difference $\delta(b) - \delta(a)$ is most frequently written in the form $\int_a^b f(x)dx$ and is called the definite integral from a to

b of $f(x)dx$. This integral may also be considered as a limit. For let the interval from a to b be divided into n equal parts $\triangle x$. Let the ordinates at the points of division be in order y_1, y_2 - - - y_n. Now consider the sum of the areas of the rectangles of the form $y_i \triangle x$:

$$y_1 \triangle x + y_2 \triangle x + \text{- - -} y_n \triangle x.$$

If $\triangle x \to 0$, this sum will approach the limit which is exactly the area under the curve $y = f(x)$ from a to b. That is to say, the limit of this sum is the definite integral $\int_a^b f(x)dx$.

I have recalled these fundamental notions of the calculus to your minds in order to point out to you that they are based on notions which come to the student as early in his mathematical career as his study of plane geometry. Thus the derivative viewed from the geometrical standpoint is the slope of a tangent which is the limit of a secant. The definite integral may be considered as the limit of the sum of the areas of certain rectangles, a notion that is no more difficult to comprehend than that of thinking of the area of a circle as being the limit of the area of a variable inscribed polygon as the sides approach 0.

How early should the calculus be introduced? One of the pioneers, and probably the most influential, in originating the present movement to socialize mathematics was the English professor of engineering, John Perry. In his famous Glasgow address[1] he made the following statement:

"In these days all men ought to study Natural Science. Such study is practically impossible without a knowledge of higher mathematical methods than that of the mere housekeeper. It must be more than what is called knowledge; it must be mental dexterity, and it must be kept in constant practice if it is not to become rusty, and if men are to remain unafraid of mathematics. As examples of methods necessary even in the most elementary study of nature I mention: the use of logarithms in computation; knowledge of and power to manipulate mathematical formulae; the use of squared paper; the methods of the calculus. Dexterity in all of these is easily learnt by all young boys."

[1] British Association meeting at Glasgow, 1901. Discussion on the Teaching of Mathematics, pp. 14, 16.

In speaking of the needs of the engineer he goes on to say: "Furthermore, although he feels his needs, bitterly, deeply, he scorns the idea that mathematics will help him in any way, for he has already done at school what was called mathematics. Such mathematics as he was taught is indeed a very useless thing to him. He wants to be able to use such very simple mental tools as the ideas of the calculus; he needs them in one way or another in every kind of engineering; he needs to be so familiar with them that he can use them in every kind of new problem that comes before him. There is no part of engineering of which the theory does not need the use of these simple tools. And we tell him that he cannot use these tools until he has worked for many years on the studies of the Alexandrian philosophers and their followers."

Perry says that the calculus can be taught to young boys. To some of us, who have in mind the sort of mathematics offered in our secondary schools and the fact that the calculus until recently has almost never been attempted before the second year of our college courses, this statement may seem incredible. That the calculus can be taught to students much younger than the students of our second year college classes may be verified by examining the courses in mathematics of England and Europe.

From a report by C. S. Jackson entitled "Calculus as a School Subject" and published in a volume entitled "The Teaching of Mathematics in the United Kingdom" we find that in Great Britain the students of calculus fall into three classes.

A. Boys of ages 17 to 19 years who compete for university scholarships.

B. Boys of ages 12 to 14 years who study the elements of the subject.

C. Boys of ages 16 to 18 years who have reached the higher mathematical classes and take calculus as a regular item of school mathematics. This class includes candidates for the army and the second division of the Civil Service.

In commenting on class B he remarks that inquiry by several teachers of experience appears to indicate that, according to the opinion of a majority of teachers, an ordinary boy is definitely unable to take in the ideas of the calculus until about the age of 15.

To exhibit the nature of problems set for Junior Appointments in the Civil Service he appends a number of examples taken from the Civil Service examination of December, 1911. Three of these follow.

1. Transform the integral $\int_{-1}^{1} f(x)\sqrt{1-x^2}dx$ by the substitution $x = \sin\theta$ and evaluate it when $f(x) = x^2$.

2. A strip of sheet metal one foot wide is to be bent into the form of a rain-water gutter of triangular section, one side of the triangle being vertical and attached to the wall of the house and the other side projecting outwards and upwards, while the third side of the triangle is open and horizontal. Find, by direct calculation, the angles and sides of the triangle when the area of the cross section of the gutter is a maximum.

3. A tube of thin glass the area of whose cross section is k, and whose length is l, has its upper end closed and it is lowered into a cylindrical jar partly filled with mercury, till it just touches the liquid. It is then depressed through a distance s. Supposing the area of a cross section of the jar to be K, and the height of the mercury barometer to be h, find equations giving the distances of the mercury surfaces inside and outside the tube below and above their original level.

Take $h = 1 = 76$ cm., $3 = 38$ cm., $k = 1$ sq. cm., $K = 4$ sq. cm. Obtain numerical results, and find the work done in displacing the mercury, neglecting the work done in compressing the air in the tube, and assuming 1 c.c. of mercury to weigh 13.6 gm.

In some countries of Europe the calculus is introduced during the eleventh year and in nearly all it forms a part of the course during the twelfth year. In France pupils of the ninth year in the science courses apply the notion of derivative to simple numerical problems and to functions previously studied.[1]

Program preceding the calculus. Accepting the view that the calculus is desirable and attainable for the boy or girl of average ability of the third or fourth year of the high school, how can a program be drawn up which will lay a suitable foundation for its study? Our present program is not properly organized for

[1] For a brief statement of the status of mathematical education in Europe see Curricula in Mathematics, United States Bureau of Education Bulletin, No. 45, 1914, by J. C. Brown.

this purpose. It is rather designed for the student who wishes to pursue a course in the theory of equations.

Since the calculus deals with related varying quantities it would seem to be desirable to develop the function concept as early as possible. In some countries of Europe the development of this concept is started in the sixth year of the elementary schools and in almost all the beginning of its development is not delayed later than the seventh year. Attention to the relationship beween quantities is obained by means of graphs and formulas.

In this country the pupil's attention is not directed to relationships existing between varying quantities even in the high school. We cannot hope for much to be done in the elementary schools along this line for many years to come but as a result of the recommendation of the National Committee we should expect that attention will be given soon to this matter in the secondary schools.

The notion of rate is one to which the pupil's attention may be called repeatedly in the junior high school. Constant rates occur in connection with such things as simple interest, uniform motion, and velocity under constant acceleration.

The pupil should see that rate means the amount of increase in the dependent variable per unit increase in the independent variable. In the case of a constant rate it is easy to show him that from the graphical standpoint rate means slope of a line. Thus let $y = mx + b$, and let c mean unit increase in x. Then for $x = c$ we have $y_1 = mc + b$ and for $x = 2c$ we have $y_2 = 2mc + b$. Thus the increase in y is mc and the rate of increase in y with respect to x is m. But m is the slope of the graph of $y = mx + b$. Preparation for the notion of instantaneous rate may be made by the study of average rates. Concrete examples can be found in connection with growth, motion, in fact, in any situation involving two variables one of which is a continuous function of the other. Thus let $s = 96t - 16t^2$ be the height in feet of a ball projected vertically after t seconds. Find the approximate velocity of the ball after 2 seconds. Let us consider a sequence of time intervals with the common end value 2 and starting with the interval from 1 to 2 as indicated in the following table. Let t mean the difference between the given value

of t and 2 and s the corresponding difference in height. Then $\frac{\triangle s}{\triangle t}$ means the average rate.

t	2	1	$1\frac{1}{2}$	$1\frac{3}{4}$	$1\frac{15}{16}$	$1\frac{31}{32}$	$1\frac{63}{64}$
s	128	80	108	119	$123\frac{3}{4}$	$126\frac{63}{64}$	$127\frac{127}{256}$
$\triangle$t		2	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$
$\triangle$s		48	20	9	$2\frac{1}{16}$	$1\frac{1}{64}$	$\frac{1}{256}$
$\frac{\triangle s}{\triangle t}$		48	40	36	33	$32\frac{1}{2}$	$32\frac{1}{4}$

From such a table the pupil sees that the rate at 2 may be approximated to any degree of accuracy desired. Incidentally he sees that the average rate is approaching 32 as a *limit*.[1]

Since the student of the calculus is required to substitute in the numerous formulas he derives, the ability to make substitutions should be cultivated from the beginning of his course in algebra. Not only should he be trained to substitute numerical values for the letters but also functions. To illustrate, suppose a stone is thrown horizontally from a cliff of height s with a horizontal velocity h to find the equation of its path knowing that

$$d = ht$$
$$s = 16t^2$$

The required equation, of course, is obtained by substituting d/h, the value of t in the first, for t in the second.

Economizing of time. How can time be found in the high school for a course in the calculus that is worth the giving? One answer to this question is that we can omit from the courses given at present much irrelevant material.

"The professional mathematician, interested chiefly in perfecting the technique of his subject, finds it natural as well as most effective to take a special group of allied methods or allied problems and to develop them as far as he can without concerning himself too greatly about the practical value of his work. Now this systematic exploration of special parts of mathematics is, no doubt, of vital importance for the continued growth of the science. It does not, however, by any means follow that the

[1] For an excellent treatment of the fundamental ideas of the calculus in the algebra course see T. P. Nunn, "The Teaching of Algebra," together with the two volumes of "Exercises in Algebra," Longmans. Also see F. L. Griffin, "Introduction to Mathematical Analysis," Houghton Mifflin Co.

branches of mathematics should be presented to beginners with the formal elaboration which is the inevitable mark of their treatment as separate subjects.''[1]

If the high school course is made homogeneous by being organized about some vital principle, such as the function concept, then each topic bears upon following topics in such a way that there is constant repetition of principles and technique. Thus, much algebraic and trigonometric drill will be carried out during the study of the calculus. Moreover such drill will be motivated. Numerous examples could be offered as illustrations of this statement. Thus, practice in solving systems of linear equations could be given in connection with the derivation of formulas. The value of the binomial theorem could be seen and practice in using it could be had in deriving the formula $\frac{dx^n}{dx} = nx^{n-1}$. Deriving the formula $\frac{d\frac{a}{x}}{dx} = -\frac{a}{x^2}$ gives practice in adding fractions.

Saving of time may be accomplished by using the methods of the calculus. Thus the mensuration of all the solids found in solid geometry can be effected by the formula $v = \int A_s dx$, where A_s is the area of a cross-section.

Amount of the calculus to be included. A high school course in the calculus must be elementary. The beginning of the course might concern itself with the differentiation of simple algebraic functions as applied to numerous problems in geometry, physics, mechanics, and various other fields. Such general problems as the following may be considered.

1. To find the slope of a curve.
2. To find an instantaneous rate.
3. To determine whether a function is increasing or decreasing for any particular value of the independent variable.
4. To find an approximate increment.
5. To determine an extreme value.
6. To find a rate given a related rate.
7. To construct the graph of a function.

The integration of the simple algebraic functions should follow their differentiation. As in the case of differentiation

[1] T. P. Nunn, "The Teaching of Algebra," loc. cit.

numerous applications can be made. Thus areas, volumes, lengths of curves, and surfaces of revolution may be computed. Interesting problems from physics, such as distance traveled at varying speed, and work done by a variable force, may be solved. The course may also include the differentiation and integration of simple exponential and trigonometric functions with many applications.

Summary. The fields of science, engineering, and industry are demanding more and more the use of the calculus in solving their problems. It is therefore desirable that our young people who are entering these fields or who wish to understand and appreciate how scientific and technical people employ the calculus should be given an opportunity to study it as early in their mathematical careers as possible. After a brief examination of the fundamental notions of the calculus we see that they are notions which are cultivated in the junior high school. Moreover an examination of the courses of study of European schools discloses the fact that courses in the calculus are offered as early as the tenth school year. It therefore seems desirable to place the calculus in the curriculum of our senior high schools. Time could be found for such a course if much irrelevant material offered in our present courses in mathematics were discarded and if these courses were organized about a fundamental principle such as the function concept in such a way that principles or topics once introduced could be kept working. Moreover the calculus itself could be used in economizing the time of the student by placing in his hands more powerful methods of investigation than those he now enjoys. The material to be included in the course must of necessity be simple. It should consist of the differentiation and integration of the simple algebraic, exponential, and trigonometric functions together with simple applications to various fields of endeavor.

TIME IS MONEY

A Story for High School and College Students and for All Who Are Interested in Saving Time

By WILLIAM E. BRECKENRIDGE
Columbia University

(Broadcasted on WJY, New York City, June 12, 1923)

Bill and Jack are two high school boys and both are members of the baseball team. Jack is the star pitcher for the school and Bill plays third base. A few days ago after school these boys met and the following conversation took place:

"I hear you have another raise in salary, Jack; you certainly are in luck."

"No luck about it, Bill. I'll tell you how it happened. You know I've been working after school three days a week in the office of an electrical manufacturing firm. Yesterday while I was in the office, our firm had a call on the 'phone from a contractor asking for an estimate on a big order of goods running into thousands of dollars. The contractor wanted an answer while he held the wire or he would give the order to some one else. My boss didn't know what to do about it. He couldn't figure out a reasonable estimate in so short a time and was going to tell the contractor so. But I thought of my *slide* rule and I said 'We can do it. Just give me the price of each article and the number to be ordered. My *slide* rule will do the rest.'

"Well, in less time than it takes to tell the story, we had the estimate ready and had closed the contract.

"You see the *slide* rule did all of the multiplication instantly. If it hadn't been for that, we would have lost the order. Of course my boss was pleased and that's how I happened to get a raise in salary."

"Good work, Jack, but tell me, what is the *slide* rule like? I have never seen one."

"The *slide* rule is just a little piece of wood about a foot long that can easily be carried in your pocket. It looks something like a twelve-inch ruler.

"But come out on the field, Bill; I want to show you a new out-drop I've been practicing and am going to use in our next

game against Eastern High School. It certainly is a winner."

"Nothing doing. I have twenty examples in physics to do for home work tomorrow and believe me, they are some work."

"I have the same lesson, Bill, but it is all done. I just ran those problems off on my *slide* rule. You know almost every problem in physics comes out in a proportion where you have to multiply two numbers together and divide by a third number. The *slide* rule does all this *for* you and gives you the answer to any proportion in a few seconds.

"And speaking of baseball, our scorer fixes up the batting average of every man on the team by using this same little instrument. You see all you have to do is to divide the number of hits by the number of times at bat and get the result correct to three figures. This is done instantly on the *slide* rule. It certainly is the quickest calculating instrument I have ever seen.

"I'll tell you another thing, Bill; I use this rule in my trigonometry home work. It helps a fellow to understand the subject. I never understood what it meant to add logarithms till I saw it done on my *slide* rule. You see after solving a triangle by the use of tables, I can find out right away by the rule whether the answers are correct. If not correct, the *slide* rule tells me where the error is. This is why I always get my work right and have it done in half the time it takes the rest of you fellows."

"Say, Jack, that instrument must be what my sister was talking about last night. You know she is private secretary to the president of the Northern Electric Company. She said she checked the reports that came to her office yesterday in one-third of the time that it usually took and she did it with a *slide* rule.

"But how about using this rule in the N. Y. State Regents' examinations, can we do that, Jack?"

"We certainly can. At the top of an examination paper that was given last year it says, 'The use of the *slide* rule is allowed for checking.' "

"Here comes my big brother, Henry. Say, Henry, what do you know about the *slide* rule? Did you ever use it while you were in the war?"

"The *slide* rule. I should say so. Do you know that if it hadn't been for the war, you fellows would probably never have

heard about this instrument. I learned to use the *slide* rule while recovering from injuries in a Government hospital at Camp Grant. While lying in bed, it was fun to fool around with the rule and see what a lot of calculations I could do in quick time. When the war closed, some people thought that an instrument that could be taught to soldiers in the hospital and that would save so much time ought to be used in the schools. So it was recommended for high schools by a national committee and that is how *you* come to know about it. I hear that it is coming into use in the regular mathematics classes in high schools and colleges as fast as teachers and students learn how interesting it is and how much time it saves. Why, even your kid brother, Tom, came home from his junior high school today and said his class had been using the *slide* rule. What do you think of that, Bill?"

"I'll say its mighty interesting, Henry. Wouldn't it be great if our school should teach us more about the *slide* rule? Look at the home work we have and think of the time we would save in our mathematics and physics. Don't you say so, Jack?"

"That's all right, Bill, but haven't you heard the news? The principal has ordered some *slide* rules for the school. Everyone in the commercial department, in trigonometry, and in intermediate algebra is going to be taught how to use the rule. As for *me*, my dad gave me a *slide* rule for my birthday. A self-teaching manual that anyone can understand came with it. I tried it out and found that I could work every example in the book. So *I* worked up the use of the slide rule by myself."

"That sounds good to me, Jack, and I'll tell you what *I'm* going to do. Tomorrow morning, bright and early, you will find me the owner of a *slide* rule like yours."

If any of the radio audience have been interested in this story of the slide rule as one of the new features in education; if you would like to hear more about it, or if you would like to ask for further information about its use, we shall appreciate it very much if you will write to Mr. W. E. Breckenridge, care of WJY, Aeolian Hall, West 42nd Street, New York City. He will be glad to answer your questions.

PUBLIC HONORS FOR MATHEMATICAL CONTRIBUTIONS

By PROFESSOR G. A. MILLER
University of Illinois, Urbana, Ill.

The student who is considering the question of preparing himself for a mathematical career is naturally interested in the possible public honors for important contributions along this line. He is probably aware that the Ph.D. degree is supposed to imply such contributions, but he may not have heard much about the public honors which normally follow this degree in case a man continues to make contributions of high order. It is evident that society should be so organized as to encourage such contributions for after the start has been made under the inspiration of the Ph.D. degree still better results may normally be expected from the more mature later efforts if they are sufficiently strenuous and prolonged.

The fact that some high public honors normally come late in life may be illustrated by the average ages of the presidents of the American and of the British Association for the Advancement of Science. In his address as president of the former, L. O. Howard noted that during the preceding quarter of a century the average age of the presidents of this association at the time of their official address was 61 years and five months, while the average age of the presidents of the British Association during the same period was 61 years and 11 months. Several of these officers were [L. O. Howard, *Science*, vol. 54 (1922), p. 640] considerably younger than this average age and a few were much older.

While this office implies very high distinction it does not necessarily imply high attainments as regards research in a particular field of science. Such distinction is more generally made the basis of election to membership in the National Academy of Sciences, which was organized in 1863 with a membership of fifty for the purpose of furnishing reliable scientific information to our Federal Government officials. In view of the rapidly increasing number of scientific investigations in our country during the last sixty years the upper limit of this membership has been increased from time to time and stands now

at 250. It is evident that this number is still sufficiently small to insure very high attainments on the part of available scientists. In pariculer, there are now only 15 members in the section of mathematics, and no other scientific body in America to which mathematicians are eligible has continually maintained such high standards as regards its membership.

Among the other general American organizations which include mathematics and select their members with such great care that it is a distinct honor to be invited to join, are the American Academy of Arts and Sciences, incorporated in 1780 and established in a permanent home provided for it in Boston by Alexander Agassiz, and the American Philosophical Society, organized at Philadelphia in 1743 and reorganized after its early demise into a flourishing society attracting to its annual meetings in Philadelphia a large body of able men representing the different fields of knowledge. The publications of each of these organizations include mathematical contributions of considerable value.

While mathematicians naturally prize recognition on the part of such general organizations, this recognition normally comes later than that which they can expect from the organizations devoted to the interests of their own subjects. At the meetings of the latter they find opportunities to talk with others who are studying questions relating to the work on which they may be engaged. This often leads to mutual helpfulness and lasting friendships which would otherwise have been improbable. When the stage of publication is reached the helpful influences can be greatly extended and the consequent feeling of comradeship is usually greatly enlarged.

Few things contribute more to the happiness of a mathematician than the feeling that he has gained the confidence and respect of those working along similar lines and that by his own work he has been able to smooth the intellectual paths for others. It is true that the teacher is largely engaged in smoothing the intellectual paths of his students, but he is usually following the roads outlined by others in doing this. The fact that he can render such service by means of a path due to his own efforts is extremely gratifying, especially if he recalls that such a path is a permanent addition to the intellectual attainments of our race.

Unfortunately public honors for meritorious work have sometimes been delayed for a long time. An international critical review which has been published in Germany since 1871 under the title *Jahrbuch über die Fortschritte der Mathematik* has tended to lessen the probability of such oversights on the part of the public. The work of this review was greatly retarded by the World War but rapid progress has recently been made towards a return to normal conditions. On the whole these reviews have done much towards securing prompt public recognition of work of high order even if it was published in periodicals of comparatively small circulation.

It follows from the above that those who contribute results of real value usually do not now have to wait many years for public recognition. This recognition may come in many ways. As a rule the national mathematical societies choose their officers from among those who seem to have done most for the progress of our science. They also select such members for their special programs and as representatives at important mathematical gatherings and on important committees. Colleges and universities also seek to strengthen their department faculties by such men whenever vacancies present themselves.

It should, however, be noted that in mathematics, ability counts for much more than diligence. One short paper of real merit is more important than a dozen longer memoirs, devoted to working out details which have little bearing on other work. It is often difficult to evaluate results until they have been before the public for a sufficient time to be considered by various men working along somewhat similar lines. Hence public honors are sometimes withheld until there is ample time for verification and estimating real merit. In particular, when the Gesellschaft der Wissenschaften zur Göttingen announced in 1908 a prize for the proof of Fermat's theorem it stated explicitly that the prize would not be awarded until after at least two years had expired from the time of the publication of the proof in question in order that the mathematicians of Germany and of other countries would have an opportunity to express their opinions as regards its accuracy.

Various other prize problems are announced by the different foreign academies from time to time and the solution of such problems carries with it public honor of very unequal degrees

depending upon the nature of the problem. During the last quarter of a century several American mathematicians have secured such prizes, and doubtless more of these prizes will be won by Americans in the near future. One sometimes reads in the newspapers that a certain mathematician could solve every problem which was presented to him. As a matter of fact, the mathematicians who are acquainted with the current literature of their subject are always aware of many problems which they cannot solve. Some of these have remained unsolved for centuries.

Some of the mathematical periodicals maintain problem departments and publish solutions with the names of their authors. In our own country the *American Mathematical Monthly* has rendered conspicuous service along this line since 1894 when it was founded by B. F. Finkel. In England more than a hundred volumes containing problems and solutions from *The Educational Times* have been published, beginning in 1864. Among the earlier English periodicals which are well known on account of the mathematical problems and solutions which appeared therein is the *Ladies' Diary*. In 1817 T. Leybourn published four volumes devoted to the mathematical questions and their solutions appearing in this journal from 1704 to 1816.

Public recognition of a much higher type is furnished by election to honorary membership in well known mathematical societies. Among the leading mathematical societies which have elected such members is the one founded in London in 1865. Two years later the noted French geometrician, M. Chasles, was elected as an honorary member of this society, and in 1871 this type of members was increased by the election of E. Betti, A. Clebsch, L. Cremona, C. Hermite and O. Hesse. Since then this mathematical society has continued the practice of honoring in this way some of the foremost mathematicians of other lands, and various younger mathematical organizations have adopted this custom while others have not done so.

The main object of the present note is to direct the attention of the younger mathematicians to the fact that those who are trying to contribute to the progress of our subject may possibly be inspired by the hope of receiving public honors at every period of life. I happen to know a man (not a mathematician) who was elected as a member of the National Academy of Sci-

ences when he was 75 years old and he seemed to be just as much pleased by this honor as the younger man usually is when his doctorate thesis is published. In both cases the main inspiration was doubtless furnished by the interest in science for its own sake, but it is fortunate that society is so organized that public honors usually come sooner or later to those who contribute largely to the intellectual heritage of our race.

The mathematician usually finds it difficult to communicate to his associates the nature of the problems which engage his attention, but he may rest assured that if he succeeds in arriving at far reaching results the merits of his work will become known. The marvelous interdependence of mathematical results tends to give a significance to his results which cannot be hoped for by those working in sciences where the interdependence is less marked. The transition from isolation to close connection is very marked in our science just as public neglect is frequently followed closely by public distinction of a high and lasting order.

A DUTCH TEXT-BOOK OF 1730

By PROFESSOR LAO GENEVRA SIMONS
Hunter College, New York City

It is natural that arithmetic should have been the first mathematical subject to appear in print in the American colonies. It is, on the contrary, surprising that algebra should occupy over one-third of the space in the third book on arithmetic published in this country[1] when nearly sixty years were to elapse before the appearance of another book[2] containing any algebra. Some forty-five hundred titles[3] of publications in Pennsylvania before 1785 show many almanacs but few works on mathematics, and none containing algebra. A complete bibliography[4] of all American books up to 1792 reveals, among those on mathematics, only three which include algebra in their contents. One of the three is a Dutch text-book and while the extent of its influence was probably very limited, it has interest as the earliest and, for a long period, the only work on algebra printed here.

The title of the book is "Arithmetica of Cyffer Konst, . . . Als Mede Een kort van de Algebra"[5] [See Fig. I]. The names which appear on the title page are the names of three men who were kindred spirits in their independence of authority.

The name of John Peter Zenger is inseparable from the history of the freedom of the press. He was the second printer of New York and his newspaper was the instrument by means of which active protest was made against the tyranny of the Royal governors which eventuated in the American revolution. The trial of Zenger is significant in all history and the outcome of it was that liberty of the press which gave people in this country the right to freely criticize the conduct of public officials. Zenger's press was established in 1726, and his newspaper had its beginning in 1733. The Venema book appeared between these two dates. Venema, Goelet, and Zenger must have had much in common, and one's fancy might dwell on their discussions of affairs of government, church, and school.

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[1] The "Sumario Compendioso of Brother Juan Diez" was the earliest mathematical work published in the New World and it contained algebra. It appeared in 1556 in the country later known as Mexico. See Smith, David Eugene, "Sumario Compendioso of Juan Diez," 1921, p. 51. ff.

[2] Pike, Nicolas. "A New and Complete System of Arithmetic," 1788. Contains "An Introduction to Algebra."

[3] Hildeburn, Charles R. "A Century of Printing. The Issues of the Press in Pennsylvania 1685-1784." 1885.

[4] Evans, Charles. "American Bibliography. 1639-1820." Complete through 1792.

[5] Only one copy has come to light. This is in the library of the New York Historical Society.

Jacob Goelet appears in a minor way as breaking away from the authority of the church. He is referred to as expressing himself in opposition to a stand taken by the ecclesiastical body in several instances. One of these was connected with the licensing of private school teachers. At any rate he succeeded

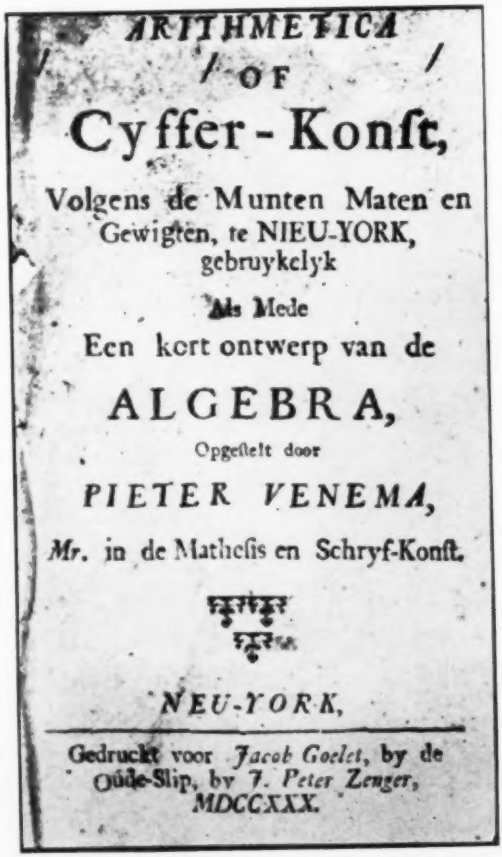

ARITHMETICA
OF
Cyffer-Konſt,
Volgens de Munten Maten en
Gewigten, te NIEU-YORK,
gebruykelyk
Als Mede
Een kort ontwerp van de
ALGEBRA,
Opgeſtelt door
PIETER VENEMA,
Mr. in de Matheſis en Schryf-Konſt.
NEU-YORK,
Gedruckt voor Jacob Goelet, by de
Oude-Slip, by J. Peter Zenger,
MDCCXXX.

Fig. I. "Arithmetic or the art of ciphering, according to the coins, measures and weights of New York, together with a short treatise on algebra drawn up by Pieter Venema, master in mathematics and the art of writing. New York, printed by Jacob Goelet, near the Old Slip, by J. Peter Zenger, 1730." The title page of the first book containing algebra that was printed in the American colonies.

[6] Ecclesiastical Records of the State of New York. 1901-1906. Vol. IV. p. 2333.

[7] Dunshee, H. W. "History of the School of the Collegiate Reformed Dutch Church from 1633-1833." 1883, p. 38.

in having printed this text-book in Dutch by a man, who, as will appear later, was under the ban of the local church leaders.

Pieter Venema was repeating history by making trouble in the church. In a letter[8] from the Rev. Gualterus DuBois to the Rev. Classis of Amsterdam, May 14, 1741, we find this complaint:

"Inasmuch as the Rev. Consistory of New York several years ago exhorted their ministers to be on their guard, and oppose the artful misleadings of one Pieter Venema, a crafty free-thinker of Groeningen, who had previously been a Reader and School-master just outside that city, I, therefore, determinedly set myself against him. Under God's blessing my efforts accomplished much good, although some still adhere to him. Among these these is one Jacob Goelet, who with his conventicles, endeavors to do all possible harm to our church."

We therefore see that Venema came from a city in Holland which offered university privileges and in which he had been a school teacher. As was so frequently the case in the early schools of New York, he may have combined preaching with teaching.

More definite evidence that Venema was a schoolmaster is given in the dedication to a work of his published in Holland in 1714.[9] In this dedication, which is signed Pieter Venema, appears the statement:

"Is de Gunst van U Ed. Mog. geweest dat ik eenige Jaren herwaarts in U Ed. Mog. niet min vermaarde Stad Groningen, mijn bedieninge als School-Meester hebbe waargenomen:"

The high regard in which he was held by the mathematicians of his own country and time is shown by an inscription to him in the work[10] already referred to. This inscription contains such phrases as

"The talent which God has granted you."

"We need no teacher, the book is a guide in itself,"

and ends with the words:

"We thank you, Venema, we thank you, brave teacher, give us more of your knowledge: You have won so much distinction at Groningen that it is impossible that you should be forgotten."

Venema must have been known during the eighteenth century and the early part of the nineteenth century because he is re-

[8] Ecclesiastical Records, loc. cit. p. 2756.

[9] Venema, Pieter. "Een kort en klare Onderwysinge in de Beginselen van de Algebra ofte Stel-konst." Te Groningen. 1714. Other editions Amsterdam 1730, 1756, 1768, 1783, 1794, 1803. Hereafter referred to as Venema, P.

[10] Venema, P. 1714 ibid. Note. No bibliography consulted revealed the existence of this edition but a copy was found in the collection of Mr. George A. Plimpton, to whom the writer is indebted for the privilege of examining the book. Mr. Plimpton also has a copy of the edition of 1756.

ferred to repeatedly in collections of problems solved and published by Dutch mathematicians and societies of that period.[11]

Venema's reasons for writing the book under consideration appear in the "Konst Lievende Leser." He says:

"Because I realized that there was here no ciphering book in the Low Dutch concerning trade or merchandise, and for the sake of the teaching of inquiring youth and of all lovers of the teaching of Arithmetic, I have undertaken to make a clear and succinct ciphering book upon that excellent science which flourishes in this city and country. To this are added the elements of algebra, whereby that which is not understood in arithmetic can be demonstrated by the clear words of algebra, so that algebra is the opening to the obscure propositions of arithmetic. Should any one desire to go farther than the subjects included in this book, he can make use of my simple Algebra or Stel-konst published in the year 1714, in my native city, Groeningen.[12]

If I observe that my zeal works out to the profit of my fellow-beings, acceptable and profitable to you people, as I anticipate, I shall likewise demonstrate parts of the algebra of Euclid; being the foundation of geometry, trigonometry, astronomy, the art of surveying, navigation and fortification which has hitherto remained unknown to my Fatherland."

The book consists of one hundred and twenty pages, of which seventy-five are devoted to arithmetic and forty-five to algebra.

The section relating to arithmetic starts with addition tables leading to the multiples of numbers up to 9×9. This is followed by addition, subtraction, multiplication, tables of weights and measures, the operations with money, rule of three, reduction of fractions to lowest terms, to common denominators, operations with fractions, rule of three, rule of five, rule of partnership, partnership with time, and alligation.

The second section bears the heading "Algebra ofte Stelkonst," that is, "Algebra or the art of place." The reason for the use of the Stel-konst is stated thus:

"This science is called by the word Stelkonst because that means, for the unknown, place x, y, z, the last three or more letters of the alphabet, and for the known, the letters a, b, c, d, and so forth."

The contents of the algebra text are as follows: Signs of operation, general notions, axioms, addition, subtraction, multiplication, including product of $a + b$ by $a + b$, $a - b$ by $a - b$, and $a + b$ by $a - b$, division, reduction of fractions to lowest terms and to a common denominator, addition, subtraction, mul-

[11] "Wiskunstige Verlustiging, etc." Vol. I, 1793, pp. 29, 63, 70, 79, 81, "Ontbindingen, etc." p. 213 ff. Vol. II, 1795, p. 8; "Wiskunstig mengelwerk, etc." Vol. II, 1802, p. 12; "Wiskunstige oeffeningen, etc." Vol. II, 1809, pp. 4, 23, 74; "Verzameling van Voorstellen, etc." Vol. 1, 1811, pp. 10, 35; Vol. II, 1815, p. 45.

[12] An examination of the 1714 algebra shows that the work of 1730 is an abbreviated treatment of the earlier work. Processes, examples and problems of the latter are identical with parts of the former.

tiplication and division of fractions, solution of simple equations and simultaneous equations in two unknowns, and problems, some leading to indeterminate equations of the first degree.

The signs of operation are not given until the section on algebra is reached. They are the usual signs for addition, subtraction and equality and a • which "betekent tot" ["denotes an empty space"]. No sign is given for multiplication but an explanation is made for letters following each other without signs. The sign for equality is printed with unusually long parallel lines.

The procedure with each topic is to state the general rule, work out an illustration of it, and prove the correctness of the result by numerical substitutions. Sets of examples accompany all rules. Among the features of the book are some which would not now be found.

One of these is the form for the division of a fraction. [See Fig. III.] The divisor, $\frac{cc}{aa}$, precedes the dividend $\frac{a}{c}$ and the two are separated by an exaggerated X. Another feature is the repetition of a letter for the second power, and a large figure to the right of the base is the representation of higher powers. Thus a^3 is written $a3$[13] [See Fig. III]. The lowest common multiple of several expressions is found by a method which is associated today with finding the lowest common multiple of small numbers. [See Fig. IV.]

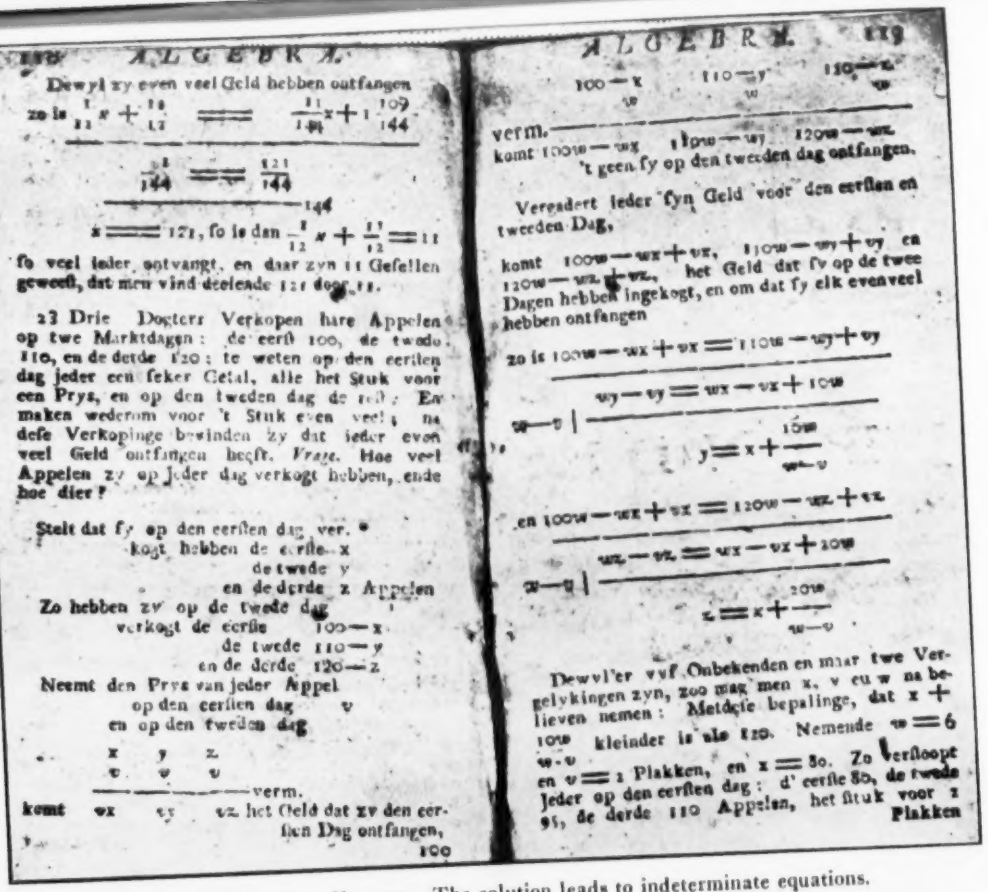
The book ends with twenty-four problems and the familiar age problem is among them. It takes the form of a curious son who asked his father his age:

"The father answered, your age with the second part, th· third part, and the fourth part of itself increased by 24½ years [is equal to mine.] I am as much over 40 years as you are under 40. How old was the son? Ans. 18 years."

Two unknown quantities are used in the solution. The problem shown [see Fig II] leads to indeterminate equations. It reads:

"Three women bought apples, the first 100, the second 110, and the third 120. They sold, each a different number, the first day, at the same price, and the remainder the second day, also at a uniform price. In

[13] This form of the exponent may have been due to the convenience of the printer, since Venema uses the present day form in his earlier algebra. [Venema P. 1714, loc. cit., p. 74], except for the second power. However, such a form of the exponent may have been known to him. It is found in Herigone, Pierre. "Cursus Mathematicus, nova brevi et clara, etc." 1644, Vol. II, Section on Algebra, p. 4, and consistently throughout the entire work.



ALGEBRA.

Dewyl xy even veel Geld hebben ontfangen

zo is $\frac{1}{12}x + \frac{18}{12} = \frac{11}{144}x + 1\frac{109}{144}$

$\frac{132}{144} = \frac{121}{144}$

144

$x = 121$, so is dan $\frac{1}{12}x + \frac{17}{12} = 11$

so veel ieder ontvangt, en daar zyn 11 Gesellen gweest, dat men vind deelende 121 door 11.

23 Drie Dogters Verkopen hare Appelen op twe Marktdagen: de eerst 100, de twede 110, en de derde 120: te weten op den eerlen dag jeder een seker Getal, alle het Stuk voor een Prys, en op den tweeden dag de rest: En maken wederom voor 't Stuk even veel; na dese Verkopinge bewinden zy dat ieder even veel Geld ontfangen heeft. *Vraag.* Hoe veel Appelen zy op jeder dag verkogt hebben, ende hoe dier?

Stelt dat sy op den eerlten dag verkogt hebben de eerste x
de twede y
en de derde z Appelen
Zo hebben zy op de twede dag
verkogt de eerste 100 — x
de twede 110 — y
en de derde 120 — z
Neemt den Prys van jeder Appel
op den eerlten dag v
en op den tweden dag w

x	y	z
v	v	v
w	w	w

verm.

komt vx − wx | vy − wy | vz − wz het Geld dat zy den eerlten Dag ontfangen, 100

ALGEBRA. 119

$\frac{100 - x}{w}$ $\frac{110 - y}{w}$ $\frac{120 - z}{w}$

verm.

komt $\frac{100v - vx}{w}$ $\frac{110v - vy}{w}$ $\frac{120v - vz}{w}$

't geen sy op den tweeden dag ontfangen.

Vereadert ieder syn Geld voor den eerlten en tweeden Dag,

komt $100v - wx + vx$, $110v - wy + vy$ en $120v - wz + vz$, het Geld dat sy op de twee Dagen hebben ingekogt, en om dat sy elk even veel hebben ontfangen

zo is $100v - wx + vx = 110v - wy + vy$

$vy - vy = wx - vx + 10v$

$w - v \mid$

$y = x + \frac{10v}{w - v}$

en $100v - wx + vx = 120v - wz + vz$

$wz - vz = wx - vx + 20v$

$z - v \mid$

$z = x + \frac{20v}{w - v}$

Dewyl er vyf Onbekenden en maar twe Vergelykingen zyn, zoo mag men x, v en w na believen nemen: Metdese bepalinge, dat $x + 10v$ kleinder is als 120. Nemende $w = 6$ en $v = 1$ Plakken, en $x = 80$. Zo verstoopt Jeder op den eerlten dag: d' eerste 80, de twede 96, de derde 110 Appelen, het stuk voor 1 Plakken

Fig. II. Problem from Venema. The solution leads to indeterminate equations.

counting their money, they found that they had equal amounts. How many apples were sold on each day?"

The unknowns, x, y, z, respectively, are taken for the number of apples sold the first day, v and w, respectively, for the price on the two days. By the conditions of the problem $y = x + \frac{10w}{w - v}$ and $z = x + \frac{20w}{w - v}$ are obtained. With the

ALGEBRA, 97

10. Verm. $x + \frac{a-b}{c}$ met $x - \frac{a+b}{c}$ komt $\frac{cxx - aa + bb}{cc}$

DIVISIO in 't GEBROKEN.

Algemene Regel.

Stelt de Deeler voor aan, dan vermenigvuldigt de Deelers Noemer, met het gedelde zyn Teller, komt Teller; en de Deeler zyn Teller, met het Gedeelde zyn Noemer, komt Noemer van het begeerde.

Ook mag men Teller tegen Teller, en Noemer tegen Noemer verkorten.

1 Deelt $\frac{ac}{bd}$ door $\frac{c}{d}$ komt $\frac{a}{b}$

$\frac{c}{b}$ $\quad$ $\frac{ac}{bd}$ $\quad|\quad$ $\frac{a}{b}$

2 Deelt $\frac{abd}{cc}$ door $\frac{ab}{c}$ komt $\frac{d}{c}$

3 Deelt $\frac{ab}{cc}$ door $\frac{bcc}{aac}$ komt $\frac{a3}{c3}$

$b,$ $\quad$ $\frac{bcc}{aac}$ $\quad$ $\frac{aa}{cc}$

$c,$ $\quad$ $\quad$ $\quad$

of $\frac{cc}{aa} \times \frac{a}{c} \quad|\quad \frac{a3}{c3}$

K $\quad$ 4 Deelt

Fig. III. Division of fractions from Venema. The word for fraction is the Dutch word "broken," for numerator, "numberer," and for denominator, "namer."

usual ingenuity in such problems, w, v, and x are taken so as to give one set of values for x, y, and z.

As simple as all this work seems there is good stuff in it. The question arises as to the schools in which it was used, for there

must have been a definite reason for printing it with the arithmetic. It is to be noted in the preface by the author that he looked upon the algebra as necessary to the clearing up of doubtful points in the arithmetic. But he, as a practical schoolmaster, must have known that the book was needed for instruction. Did Venema himself have a private school and did he succeed, so

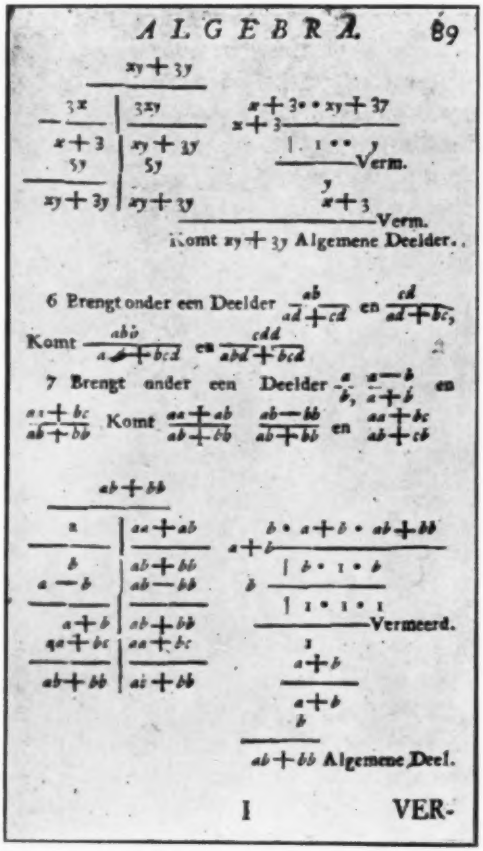


Fig. IV. Reduction of fractions to a common denominator from Venema.

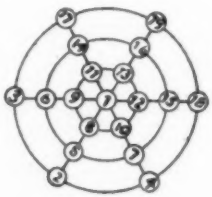
early in the history of printing in this country, in putting into print the material that he needed? It seems to be a safe conjecture to put him at the head of one of these schools which were common enough in New York at this period. And the algebra of at least one secondary school of that period was not unworthy.

MAGIC CIRCLES

By VERA SANFORD
The Lincoln School

The *Algebraic Magic Squares* of Mr. McLaughlin's article (Mathematics Teacher, February, 1921) have a natural corollary in the magic circles of Japanese Mathematics.

In these figures n concentric circles are cut by n diameters, and numbers are placed at the points of intersection in such a way that the sum along any diameter or along any circle is a constant. According to the illustrations in Smith and Mikami's *History of Japanese Mathematics*, it was customary to put the number 1 in the center. Thus a magic circle given by Isomura (1660) appears as follows:



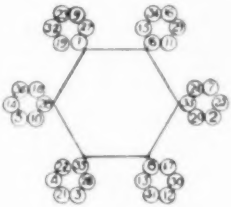
Therefore, while the magic square permits the use of n^2 numbers only, the magic circle takes care of $2n^2 + 1$.

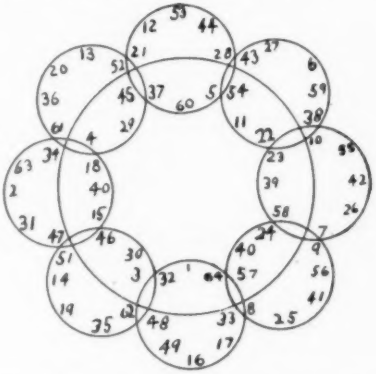
The problem of building magic squares and circles interested the greatest Japanese mathematicians of the 17th century. Isomura, for example, varied the subject by introducing magic wheels which take several forms. In one of these the rows of a magic square are bent into circles and are placed at the vertices of a regular polygon. Thus the sum of the numbers in any one circle equals the sum of the numbers in corresponding places in all the circles. Another form is the simpler one of interlacing circles where, apparently, only the sum of the numbers in the several circles was to be a constant.

Even with arabic numerals, the magic circle is more satisfying to the eye than the magic square, but this difference is much more evident when the magic circle appears with its original characters as in the illustration from Muramatsu[1]. This Mura-

[1] Smith and Mikami, p. 79.

matsu, by the way, was one of the retainers of the Lord Asano upon whose story Masefield based his play *The Faithful.*





The subject of magic squares interested Seki Kawa who properly ranks as the Newton of Japan, and whose work on determinants was in advance of that in Europe at about the same time. Seki laid down a rule for the formation of magic circles which might be given in this general form:

Call the intersections of any diameter with any one of the circles corresponding points. Place the numbers at will on half of these points. Then continue, taking the corresponding points in the reverse order.

It does not seem to occur to these mathematicians that any arithmetic progression would work as well, or that, if one wished to multiply the numbers, a geometric progression would serve the purpose. This suggests that we might use magic circles as motivation for arithmetic and geometric progressions. But this is a long way to show that the sum of the k-th term from the beginning and the k-th term from the end is a constant in the one case, and that the product is a constant in the other.

It is not unlikely that the hold that these magic squares and circles have upon us is due to the fact that they arouse our curiosity while their innate uselessness interests that rather desultory section of our minds that welcomes a useless thing just because it cannot possibly be practical. So the magic circle slips into that group of oddities that we term Mathematical Recreations, the serious problems of a bygone age that are sometimes drawn from their resting place to be the amusement of a day.

SYSTEMATIC PROCEDURE IN THE SOLUTION OF ALGEBRAIC PROBLEMS[1]

By ROBERT R. GOFF
High School, New Britain, Conn.

The usual directions for solving problems are:

1. Represent each unknown number algebraically.

2. Translate the unused statements of the problem into equations.

3. Solve the equations.

But high school students need more than this. They need to be led through a little analysis, classification, and determination of method. Much more emphasis, also, should be placed upon the group or class of problems and less upon the particular problem.

Now a study of many problems shows a natural division into three groups or cases, each group determined by the nature of the unknown numbers. If, then, this division is made, the distinctive step in each case will be the representation of the unknowns. Afterwards, the second and third steps will be the same in all cases.

Case I. The first kind of problem usually has two unknown numbers, the second of which can be derived from the first.

Models

1. The sum of two numbers is 63 and their difference is 17. Find the numbers.

Solution:

[2]One number is x
Other number is $63 - x$

$$(63 - x) + x = 17$$

The equation is made from an unused statement of the problem.

Occasionally there will be a third unknown also derived from the first. Thus:

2. Find three consecutive numbers whose sum is 39.

[1] Taken from the writer's Revised Outlines in Algebra. The Palmer Co., Boston.

[2] The natural order of translation is from English into algebra.

Solution:

First number is	x
Second number is	$x+1$
Third number is	$x+2$

$$3x+3=39$$

Sometimes the equation can be made from a formula of geometry or physics, suggested by the problem:

3. The perimeter of a rectangle is 288, and the length is twice the width. Find the length and width.

Solution: The formula for perimeter is $p=2l+2w$.

Width is	x
Length is	$2x$

$$2x+4x=288$$

Problems of digits, the clock, and the lever come under case I.

Case II. The second kind of problem usually has two unknown numbers like case I, but in addition to these, it has two or more unknown numbers, each derived from one of the first two by statements of the problem.

Models

1. The sides of a rectangle are in the ratio of 5 to 2. If 2 inches are added to each side, the ratio is 4 to 3. Find the sides. (Coll. Bd.)

Solution:	First time	Second time
First side is	$5x$	$5x+2$
Second side is	$2x$	$2x+2$

$$\frac{5x+2}{2x+2}=\frac{4}{3}$$

There are always two columns: 1. First time; 2. Second time. The second column is derived from the first column by statements of the problem. The equation is usually made with the derived unknowns.

2. A is 3 times as old as B but in 10 years he will be only twice as old. Find their ages:

Solution:	First time	Second time
A's age is	$3x$	$3x+10$
B's age is	x	$x+10$

$$3x+10=2(x+10)$$

Occasionally there will be more than two simple unknowns or more than two derived unknowns:

3. If A gives B \$6, B will have $\frac{2}{3}$ as much as A has left; but if B gives A \$5, B will have $\frac{3}{5}$ as much as A then has. How much had each? (Coll. Bd.)

Solution:	First time	Second time
A's amount	x	$x-6$
B's amount	y	$y+6$
		$y+6=\frac{2}{3}(x-6)$
A's amount	x	$x+5$
B's amount	y	$y-5$
		$y-5=\frac{3}{5}(x+5)$

Here the two simple unknowns are not closely related, and consequently two general numbers should be used and two equations made.

It is of the utmost importance that the work in all problems be arranged systematically as in the models.

Case III. The third kind of problem has four unknown numbers like case II, but the derived unknowns are not obtained from a statement of the problem but from an implied relationship. For example, we know that distance equals rate multiplied by the time although it may not be stated in the problem. Eight types are considered here:

Type	Important Numbers to be Represnted		
1. Areas	Length	Width	Total value
2. Number and price	Number	Price	Interest
3. Interest	Principal	Rate %	Time
4. Uniform motion	Time	Rate %	Rate Distance
5. Joint work	Time	Rate of loss	% of work
6. Weight in water	Weight	%, quantity	Total loss
7. Mixtures	Am't, mixture	Specific gravity of mixture	Am't, quanity
8. Specific gravity	Weight of equal amount of water		Weight of mixture

The product of the first two numbers equals the third. The formula is $ax=y$. This gives the derived unknowns. Hence it is very important that the three parts in each type be well learned.

Models

1. If the base of a certain square is increased 15 feet, and its altitude decreased 2 feet, the area of the resulting rectangle will be twice the area of the square. Find the sides of the square. (Regents.)

Solution:	Length	Width	Area
Square	x	x	x^2
Rectangle	$x+15$	$x-2$	$(x+15)(x-2)$

$$(x+15)(x-2) = 2x^2$$

There are three columns and the equation is made from the derived unknowns which form the last column filled in.

2. A man bought a certain number of oranges for \$2. If he had paid 5 cents more per dozen, he would have received two dozen less for the same money. How much did he pay per dozen? (Coll. Bd.)

Solution:	Number	Price	Total value
First time	$\frac{200}{x}$	x	200
Second time	$\frac{200}{x+5}$	$x+5$	200

$$\frac{200}{x} = \frac{200}{x+5} + 2$$

It will be noticed from the previous model that the simple unknowns are not always in the first column nor the derived unknowns in the third column.

3. \$10,000 is invested in two parts, the first part at 6%, the second part at 5%. The total annual income is \$580. Find the amount of each investment.

Solution:	Principal	Rate	Interest
First part	x	.06	$.06x$
Second part	$10{,}000 - x$	.05	$500 - .05x$

$$500 - .05x + .06x = 580$$

4. A man starts from his home to catch a train, walking at the rate of one yard in one second, and arrives 2 minutes late. If he had walked at the rate of 4 yards in 3 seconds, he would have arrived 2½ minutes early. Find the distance from his house to the station. (Coll. Bd.)

Solution:	Time	Rate	Distance
First time	x	1	x
Second time	$\frac{3x}{4}$	$\frac{4}{3}$	x

$$x = \frac{3x}{4} + 270$$

5. A and B can do a piece of work in 10 days, but at the end of 7 days A stops working and B finishes by working alone for 5 days. How long would it take each man to do the entire piece of work alone? (Coll. Bd.)

Solution:

Let the number of days in which A can do the work be x.
Let the number of days in which B can do the work be y.

	No. of days	Rate per day	Per cent of work
A	10	$\frac{1}{x}$	$\frac{10}{x}$
B	10	$\frac{1}{y}$	$\frac{10}{y}$
			$\frac{10}{x}+\frac{10}{y}=1$
A	7	$\frac{1}{x}$	$\frac{7}{x}$
B	12	$\frac{1}{y}$	$\frac{12}{y}$
			$\frac{7}{x}+\frac{12}{y}=1$

The formula is $\frac{a}{x}+\frac{b}{y}=1$

7. A photographer has two bottles of diluted developer. In one bottle 10% of the contents is developer, the rest water; in the other, the mixture is half and half. How much must be drawn from each bottle to make 8 oz. of mixture in which 25% is developer? (Coll. Bd.)

Solution:

	Amount of mixture	Per cent of developer	Amount of developer
First mixture	x	.10	$.10x$
Second mixture	$8-x$	.50	$4-.50x$
Third mixture	8	.25	2
			$.10x+4-.50x=2$

The formula is $an+bn^1=c(n+n^1)$

Summary

Classification:

There are three common cases of verbal problems.

I. One unknown.

One or more other unknowns derived from the first unknown by statements of the problem.

II. Two simple unknowns like case I.

Two other unknowns, each derived from one of the first two by statements of the problem.

III. Two simple unknowns like case I.

Two other unknowns, each derived from one of the first two by an implied relationship whose formula is $ax = y$.

(Cases II and III are sometimes extended to three or more simple unknowns or three or more derived unknowns.)

Representing the unknowns:

Case		
I		1. In one column represent two simple unknowns with one general number if possible.
II		2. Fill in a second column from statements of the problem.
III		3. Derive the remaining column by the formula $ax = y$.

If the first two unknowns are not closely related, two general numbers can be used, but two equations must then be made.

Forming the equations:

In all three cases the equation is made usually with the derived unknowns from:

a. An unused statement of the problem.

b. A formula of geometry or physics suggested by the problem.

THE PRICE OF WISDOM

By GEORGE W. EVANS
Charleston High School, Boston

The present economic situation, say the commentators on the news, is disastrous especially to the "middle class." The misfortunes referred to are obscure in origin, but the causes can be fairly well seen where the conditions are exaggerated; Germany furnishes a sad and obvious diagram. The buying power of a fixed sum of money becomes less and less, and the persons who depend on an income that is not capable of numerical increase, such as men on salary or old people who have invested the savings of their busy youth, have to do the best they can under conditions increasingly severe. Still more does the endowed university or library suffer; for its expenses are enormously increased, while its income is a nearly constant percentage of an unchanged capital.

This distress comes upon all learned societies, and consequently upon all scientific research. The income of such societies is often in some degree from endowment, but mostly from subscriptions, dues, or other contributions from active scientific men. They have the same exaggerated expense, and under the prevailing narrowness of situation there is a limit to the increase of dues, and to the expansion of membership.

For subjects like chemistry and electricity great organized industries have a lively interest, and, while maintaining their own staffs of learned and energetic investigators, will probably be willing to extend effective support even to institutions that are not commercial in purpose. Mathematics is not of such wide appeal. It is true that mathematics is the groundwork even of the very practical sciences that make modern industry possible; but a popular and somewhat scornful distinction is made between these applications and the investigations that seem purely theoretical and entirely remote from industrial affairs.

It is not so long ago, however, that Maxwell was elaborating the theory of electricity; carrying to apparently purposeless refinements the mathematical expression of ideas conceived by Faraday, who was untaught in mathematics; when the dynamo was a toy, and when no grocer had a telephone. There was no

Marconi when the mathematics was developed which now is essential for wireless communication. Before the practical problem was put to us to deal with, the laws for thinking it out had been prepared by scholars who were seeking only the completion of beautiful theories.

The economic interests of the whole world owe such debts as this to purely theoretical mathematics; but there is another debt, less sordid, which we all owe for a completer knowledge of the universe. In that universe, except for our ability to think about it, we live insignificant; but by means of theoretical results, pursued at first for their own sake, we interpret the scanty light from distant stars, investigate their chemistry, their origin and destiny; we explore the interior of atoms, descry the swinging orbits of their infinitely small components, and find the fairy tales of the alchemists dimly legible; we see the riddle of old Time, of the past and the future, with a new vision, and can have some notion at last of what is meant by a Mind to which a thousand years are but as yesterday.

Scholars of other days were maintained by powerful chieftains, and repaid their patrons by their own renown and by adulation in the dedications of their books. They were bitter in rivalry, jealous and secretive. The scholars of today are teachers in universities, well earning their modest living, honored for their own sake. They are generous in the spread of wisdom, working gladly with each other, eager to welcome the promising acolyte. Great nobles of feudal days might temper hard times to distressed scholarship; the learning of today must depend upon its own workers, and upon its lovers among the general public.

American scholars, within the last generation or so, have founded a tradition of high achievement; now that the lamentable disasters in Europe threaten the future of thought in the world, they are as essential as in 1918 our soldiers were.

The American Mathematical Society, known throughout the world, is the organized expression of mathematical research in this country. It is in great need. Lack of money to publish the work of its own scholars cripples it. The publication of the Transactions is over a year behind; recently, when much valuable material had accumulated, an anonymous donor left $4,000 on the doorstep, to pay for one issue.

The Society plans to raise $25,000 among its own members, and hopes to find outside of its membership public spirited generosity enough to increase the total to $100,000, as a permanent endowment. We can help by knowing the value of mathematical research, and its precarious situation; so that when occasion arises we may bear witness.

JAPANESE PROBLEMS[1]

By SHIGE HIYAMA

A priceless collection of mostly unpublished and as yet untranslated Chinese and Japanese manuscripts is stored in a curious old treasure chest belonging to Dr. David Eugene Smith of Teachers College. The contents of one of the two Japanese manuscripts which I have recently translated is the subject of this article.

The title of this manuscript is *Sandai Shu* (*Iippiaku Mon*) which may be translated as "A Collection of One Hundred Mathematical Problems." It was copied from an older manuscript in the 14th year of Bunka (1818) by Nobujiro Hirata, who is apparently unknown as a mathematician. Neither the author nor the date of the original work is indicated.

According to the custom of that time, a mathematician answered a hundred problems of a predecessor, and formulated a hundred new problems to be solved by his followers. This *Sandai Shu* was doubtless such a collection. Judging by the nature of the problems, we might attribute them to one *Hoshuku Isiye*, who lived about 1780, though the conjecture has not a very substantial historical basis.

Aside from the principles involved and the material used as illustrations, the presentation of the exercises themselves is extremely interesting. They are all numerical calculations of geometric problems and follow no progressive order. Several questions grouped around a similar geometric fact offer the only suggestion of coordination. There are, for example, six consecutive examples of the following type:

"Given the area of a rectangle as 48 *bu*, the ratio of the sum of the two dimensions to their difference as 7, find the length. The answer is 8 *bu*." (*Bu* is a Japanese square and linear measure. The problem simply requires the solution of the equations $xy = 48$ and $x + y = 7(x - y)$. No solutions appear in the text.)

Similarly, there are about four examples of which the following is typical:

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[1] In the preparation of this interesting article on the native problems of the Japanese, Miss Hiyama was assisted by Miss Mary E. Joiner and Miss Lillian Steigman.—Editor.

"Given the sum of the areas of two squares and the sum of the sides of these squares, find the length of the side of each square."

Each of the problems is presented with an incomplete diagram. The data given are very brief. The statement to be proved is indicated merely by the answer given.

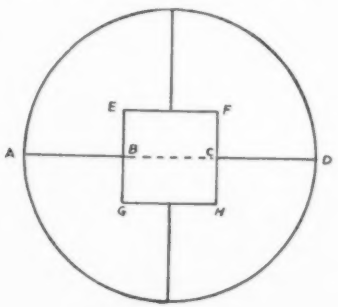
Written for mathematicians, the problems frequently require an involved algebraic equation for their solution and sometimes make it necessary to add intricate construction lines to the given figures for their proof. Various facts must also be taken for granted; for example, that given lines are segments of the same straight line; that a given line will, when extended, pass through the center of a certain circle; or that a line is a perpendicular bisector of another given line.

For these reasons the problems are not adapted to the rigorous system of a modern class in demonstrative geometry. They would, however, make highly interesting material for supplementary work.

The six diagrams with their solutions, given below, seem unusually noteworthy and characteristic examples of the *Sandai Shu*.

The area of a Japanese coin is 18.625 *bu*, and each arrow is 2 *bu*. Answer, 6 *bu*.

The figure shows the old form of Japanese coin with a square hole at the center. The "arrow" is the line AB, and refers to the position of an arrow with respect to the bow (arc). This



use of the term came from the Greek and Roman civilization; or, at any rate, it is found there. The modern solution is as follows:

Given $AB = CD = 2$ *bu.* Area of coin 18.625 *bu.* Required to find $AB + BE + EF + FC + CD$.

Let $x =$ the length of EF.

Then $x^2 =$ area of square $EFHG$,

and $x^2 + 18.625 =$ area of circle.

Therefore $\sqrt{\dfrac{18.625 + x}{\pi}} =$ radius of circle,

and $2\sqrt{\dfrac{18.625 + x^2}{\pi}} = 4 + x.$

Solving $x = 1 = EF$.

$\therefore AB + BE + EF + FC + CD = 6$ *bu. Answer.*

The area of a circle outside of a rectangle given within the circle is 65.04 *sun*; the difference between the longer and the shorter side of the rectangle is 2 *sun*, and each arrow is one *sun*. Answer: the diameter is 12 *sun*.

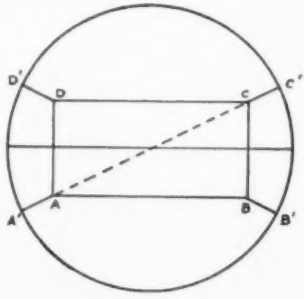
The modern solution is as follows:

Given area of the circle outside the rectangle = 65.04 *sun*.

$$AB - BC = 2 \text{ sun}$$

$$AA' = BB' = CC' = DD' = 1 \text{ sun}.$$

Required to find the length of AC.



Let $x =$ length of BC.

Then $x + 2 =$ length of AB.

$x(x + 2) =$ area of rectangle $ABCD$,

$65.04 + x(x + 2) =$ area of the circle $A'B'C'D'$.

$$\sqrt{\frac{65.04 + x(x+2)}{\pi}} = \text{diameter } A'C',$$

and $\sqrt{x^2+(x+2)^2}$ = diagonal AC of rectangle.

Hence $\sqrt{x^2+(x+2)^2}+2=$ diameter $A'C'$,

and $2\sqrt{\dfrac{65.04+x(x+2)}{\pi}}=\sqrt{x^2+(x+2)^2}+2.$

Solving, $x=6$.

$A'C'=\sqrt{x^2+(x+2)^2}+2=12$ *sun.* *Answer.*

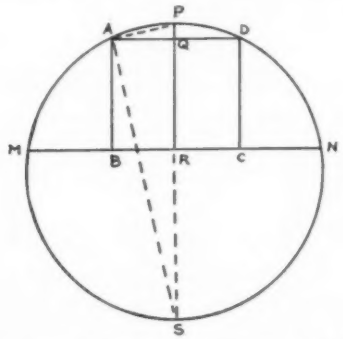
A square inscribed in the segment of a circle whose diameter is 2.5 *shiyoku*, and the arrow is 0.75 *shiyoku*. Answer, the side of the square is 0.7 *shiyoku*.

Given the square $ABCD$ inscribed in segment MPN,

the diameter of circle = 2.5 *shiyoku*,

and the arrow $PR=0.75$ *shiyoku*.

Required to find length of side AD of the square.



Complete the circle of which the segment MPN is a part. Extend PR cutting the circle at S. Draw AS and AP.

Let $x=$ side AD,

$PS=2.5$ *shiyoku*,

and $\angle PAS=$ a right angle, since PS is a diameter.

In the triangle APS we have

$$SQ:AQ=AQ:QP,$$

$$RS=2.5-0.75=1.75,$$

$$SQ=1.75+x,$$

$$QP=0.75-x,$$

$$AQ=\tfrac{1}{2}x.$$

Hence $(1.75 + x) : \frac{1}{2}x = \frac{1}{2}x : 0.75 - x$.

Solving, $x = 0.7$ *shiyoku* $= AD$. *Answer.*

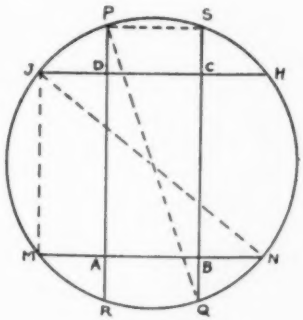
Each of the four shorter arrows is 9 *sun* and each of the longer arrows is 13 *sun*. The difference between the two sides of the rectangle formed is 16 *sun*. Answer the shorter side of the rectangle is 14 *sun*.

Given $AR = BQ = SC = PD = 9$ *sun*,

$MA = BN = HC = JD = 13$ *sun*,

and $BC - AB = 16$ *sun*.

Required to find the length of AB.



Let $x =$ length of AB,

whence $x + 16 =$ length of BC.

Draw JM, JN, PQ, and PS.

In the $\triangle JMN$, $JN = \sqrt{(x + 26)^2 + \overline{JM}^2}$

In the $\triangle PSQ$, $PQ = \sqrt{(x + 16 + 18)^2 + \overline{PS}^2}$

Since $JM = BC$, $PS = AB$, and $JN = PQ$,

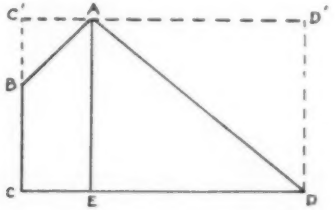
we have $\sqrt{(x + 25)^2 + (x + 16)^2} = \sqrt{(x + 34)^2 + x^2}$.

Solving, $x = 14$ *sun, answer.*

The sides of a given trapezium are 17, 5, 4 and 18 *sun* respectively.

Answer, the line drawn perpendicular to the base from the vertex is 8 *sun*.

Given $AD = 17$ *sun*, $AB = 5$ *sun*, $BC = 4$ *sun*, and $CD = 18$ *sun*. Required to find the length of AE.



Draw $AC' \parallel CD$, produce CB to meet it at C'.

Draw $DD' \parallel BC$, produce $C'A$ to meet it at D'.

Let $x = CE$, $18 - x = ED$, $\sqrt{5^2 - x^2} = BC$

$$\sqrt{5^2 - x^2} + 4 = CC' = AE,$$

and $$\sqrt{17^2 - (18 - x)^2} = AE.$$

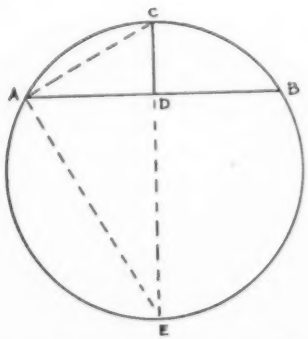
Then $$\sqrt{17^2 - (18 - x)^2} = \sqrt{25 - x^2} + 4.$$

Solving $$x = 3 \text{ } sun = CE,$$

$$18 - 3 = 15 \text{ } sun = ED,$$

and $\sqrt{17^2 - 15^2} = 8$ *sun* $= AE$, answer

The sum of the diameter of a circle and the shorter of its segments made by a chord perpendicular to it, is 11 *sun*. The product of the diameter and this chord is 60 *sun*. Answer, the diameter is 10 *sun*.



Given $CE + CD = 11$ *sun*, and $CE \cdot AB = 60$ *sun*.

Required to find the length of diameter CE.

Let $x =$ length of CD;

then $11 - 2x =$ length of ED,

$11 - x =$ length of CE,

and $\dfrac{60}{2(11-x)} =$ length of AD.

Draw AC and AE.

In the triangle ACE

$$ED : AD = AD : CD$$

$$\text{or } 11 - 2x : \frac{30}{11-x} = \frac{30}{11-x} : x.$$

Solving $x = 1$ *sun* $= CD$,

and $11 - x = 10$ *sun* $= CE$, answer.

A STUDY IN FRACTIONS

By J. E. WORTHINGTON
Waukesha, Wis.

As a result of the feeling on the part of the teachers of algebra that their students were unable to handle the simple processes in fractions, a cooperative study of the ability of the pupils in solving exercises involving fractions was decided upon. The problems that were raised and the solution of which were attempted were the following:

1. Is it true that the high school student does not know the simple processes in fractions?
2. Is he capable of solving examples in fractions, required as a foundation for algebra?
3. As far as the six-year high school is concerned, where is the improvement made in fraction-solving ability?
4. What are the types of examples most frequently missed?
5. What is the nature of the errors?
6. What remedial work can be done by the high school?

In preparing for the test a list of 25 examples was prepared. Three principles governed in this preparation.

1. Practically all types of fraction exercises were to be included.
2. An attempt was made to have examples so simple that every one would get some, and no one would get all correct.
3. The examples were arranged presumably, according to difficulty.

The following is a list of the examples finally selected and given to the pupils:

1. Reduce to whole or mixed numbers, $^{15}/_{3}$, $^{14}/_{8}$ =
2. Change to an improper fraction $11\frac{3}{7}$ =
3. Reduce to lowest terms $^{48}/_{128}$ =
4. Reduce to thirty-seconds $\frac{3}{4}$ =
5. Find the result of $27 \times \frac{7}{9}$ =
6. $\frac{2}{3}$ plus $\frac{1}{2}$ =
7. $^{11}/_{12} \div {}^{33}/_{48}$ =
8. $25 - {}^{13}/_{18}$ =
9. $42 - 15\frac{7}{12}$ =
10. $^{13}/_{88} \times 121$ =
11. Express as a fraction .003
12. $23\frac{1}{4} - 16\frac{3}{4}$ =

13. $\frac{1}{2}$ of $\frac{3}{4} \div \frac{7}{8} =$
14. $5 \div 4\frac{3}{8} =$
15. Change to similar fractions $\frac{1}{3}$, $\frac{1}{7}$, $\frac{1}{21}$
16. $6\frac{5}{6}$ plus $2\frac{1}{9}$ plus $3\frac{1}{4}$ plus $6\frac{1}{6}$ plus $2\frac{7}{18} =$
17. $76\frac{3}{4} \times 12\frac{1}{2} =$

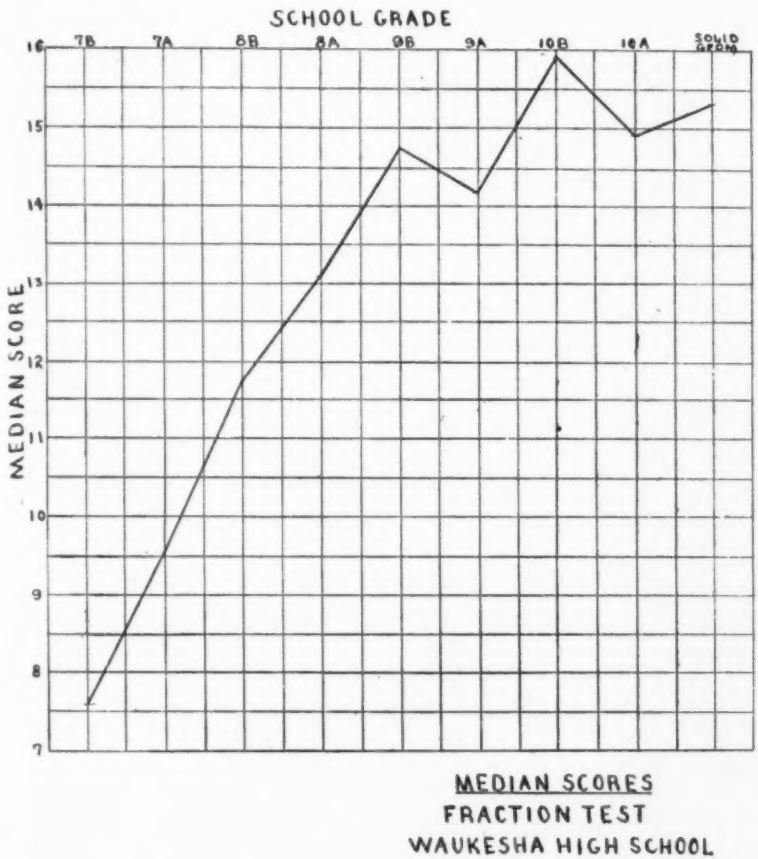


Figure 1. See Table I.

18. Reduce to a simple fraction $\dfrac{6}{8\frac{1}{2}} =$
19. $\frac{2}{3} \times \frac{9}{10} \times \frac{30}{49} \times 1\frac{2}{5} \times \frac{1}{2} \times \frac{5}{9} =$
20. $3\frac{3}{8}$ of $4\frac{4}{5} \div \frac{2}{5}$ of $6\frac{1}{2}$ of $\frac{3}{8} =$
21. $32\frac{7}{8}$ plus $91\frac{3}{4}$ plus $52.125 =$
22. $23\frac{5}{8}$ plus $23\frac{8}{9}$ plus $18\frac{3}{4}$ plus $27\frac{5}{6}$ plus $28\frac{2}{3} =$

23. $\dfrac{\frac{2}{11} \text{ of } 12\frac{6}{7}}{\frac{3}{8} \text{ of } \dfrac{5}{6\frac{2}{3}}} =$

24. $\dfrac{\frac{3}{8} + \frac{13}{16} \div 1\frac{19}{20}}{2 - 1\frac{13}{16} + \frac{8}{77} \times 1\frac{1}{20}} - \frac{19}{36} =$

This study was conducted by Mr. Worthington and the teachers of mathematics in the high school.—Editor.

The examples were given to all students of mathematics in the Waukesha High School, which includes grades 7 to 12 inclusive. As the exercises were mimeographed and as some necessary mathematical signs cannot be made on the ordinary typewriter, the teachers were instructed to explain carefully any unusual

TABLE I

No. of Problems Correctly Solved	Frequency Distribution of Marks								
	7B	7A	8B	8A	9B	9A	10B	10A	Solid Geometry
24	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	1	0	0
22	0	0	0	0	2	2	1	0	0
21	0	0	0	0	1	2	9	0	3
20	0	0	0	0	5	7	11	3	0
19	0	0	1	3	8	2	17	5	1
18	0	0	3	4	9	5	16	4	2
17	0	0	2	3	11	6	16	7	1
16	0	2	1	3	19	6	25	3	1
15	1	1	4	5	12	8	23	6	5
14	4	1	3	7	12	5	18	5	1
13	2	1	3	5	18	8	12	7	0
12	3	7	9	6	9	8	16	6	3
11	4	7	6	7	5	9	4	2	1
10	9	11	8	6	6	2	8	2	4
9	7	10	8	2	4	4	3	4	0
8	8	7	3	3	3	1	3	2	1
7	8	3	7	1	2	3	0	0	0
6	11	5	2	1	6	5	2	0	0
5	4	1	3	1	3	0	0	0	0
4	9	7	3	0	1	0	2	0	0
3	4	4	3	0	3	1	1	1	0
2	3	0	0	0	2	0	0	0	0
1	5	2	2	0	9	0	0	0	0
No.	83	69	92	57	141	84	178	57	23
Median	7.6	9.55	11.8	13.1	14.7	14.2	15.87	14.9	15.3

method of stating the examples, so that all would understand. The tests were scored by the pupils. The number correctly solved was regarded as the pupil's score.

I. Table I is a statement of results.

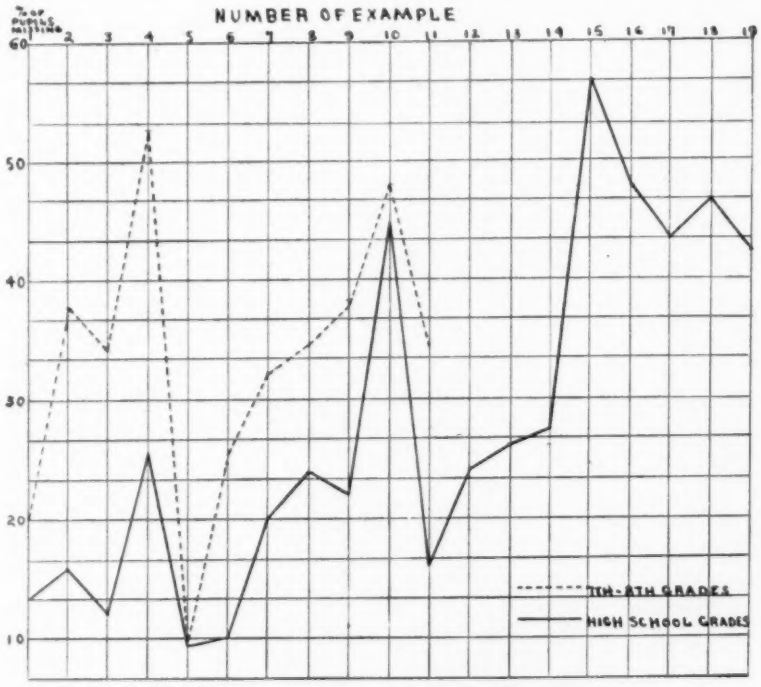


Figure 2. See Table II.

II. Next an attempt was made to find out the examples which caused the most trouble. The results of the study are found in Table II.

III. Common mistakes in the solution.

About 100 papers were selected and the solutions of each example were studied in an effort to find out the more common mistakes. The results are indicated in Table III.

TABLE II

Number of Pupils Missing Each of the Twenty-four Examples

No. Problem	SCHOOL GRADES								Solid Geom	% 7-8th Grade Students	% 9-12th Grade Students
	7B	7A	8B	8A	9B	9A	10B	10A			
1	29	21	18	15	28	8	17	11	1	27	13
2	43	31	29	12	34	15	24	4	4	38	16+
3	37	29	24	13	26	5	20	6	4	34	12+
4	48	32	49	31	50	19	42	10	5	53	25+
5	17	4	7	1	8	13	13	9	3	9	9+
6	34	22	15	6	16	8	17	6	4	25	10+
7	42	26	22	7	37	19	29	9	9	32	20+
8	46	38	32	19	30	27	47	15	5	35	24+
9	41	29	37	9	26	28	39	14	3	38	22
10	43	36	40	24	64	47	67	37	11	48	45+
11	41	27	24	11	54	18	28	4	2	34	16
12	40	33	37	14	33	29	45	11	3		24+
13	42	29	26	10	44	32	34	15	7		26+
14	40	28	32	13	43	33	42	13	8		27+
15	21	49	64	49	95	42	108	33	11		57+
16	14	36	54	38	75	42	84	29	12		48+
17	16	29	37	21	63	40	76	29	10		43+
18	8	22	58	37	71	54	79	28	5		47+
19	6	14	51	31	67	43	64	26	11		42+
20	1	5	67	53	123	72	160	48	22		85
21	0	1	66	41	83	54	126	45	15		64+
22	0	0	67	49	104	59	135	43	18		71+
23	0	0	71	54	133	79	138	52	21		84+
24	0	0	71	57	132	84	158	58	22		90+

The kind of errors in Table III are indicated by letters representing the following:

A = Mistake in number combinations or in the fundamental operations of whole numbers.

B = No knowledge of what was to be done—the pupil did not recognize the problem, or the method to use in solving it.

C = Scattering.

D = Carelessness.

E = Not reduced to lowest terms.

F = Disregarded Denominator—regarding Numerator as whole number.

G = Inverted Dividend.

H = Incorrect Cancellation.

I = $\frac{2}{3}$ plus $\frac{1}{2} = \frac{3}{5}$.

J = Mistake in reducing to similar fractions.

K = Cancelled and multiplied terms together.
L = Mistake in reducing to lowest terms.
M = Subtracted whole numbers but added fractional parts.
N = Used 100 or 10 as denominator.

Table III should read as follows: 60% of all errors made in Problem No. 1 were due to mistake in number combinations, or in the fundamental operations; 14% of all errors in Problem No. 1 were due to no knowledge of what was to be done, etc.

The greatest number of mistakes were due to errors in number combinations, or inaccuracy in the fundamental operations. The second largest number of errors was due to the pupils' inability to recognize the problem. He did not know how to begin. Most of the other errors were characteristic of a certain type of fraction example. As the problems become more complex it was increasingly difficult to analyze the mistake, the teacher usually calling it carelessness.

TABLE III

No. of Problem	Kind of Errors													
	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	60	14	17		9									
2	8	28	52			12								
3	30	10	51		9									
4		75	20			5								
5			100											
6			45						15	40				
7		14	34				40	6			6			
8	12	27	37	15								9		
9	7	19	42										32	
10	33	11	22	33										
11		17	17	22										45
12	7	4	45	33		11								
13	57	28		15										
14	73	18		9										
15	46	50		5										
16	51	49												
17	50	30		20										
18		80		20										

Suggestions for Improvement

1. In the seventh and eighth grade classes the plan suggested by one of the teachers should be followed, in correcting common

mistakes in fractions. She says, "I take a few minutes each day on some common error. The next day, I have examples or problems on the board to illustrate various errors that have been discussed in class. The students solve these during the time I am on hall duty, or taking roll, etc. We correct these, and if necessary, discuss them. Then we go on with the regular work."

2. Motivated timed drills in examples dealing with fractions should be carried on in arithmetic and algebra classes. In algebra the teacher should anticipate the pupil's trouble with fractions, and prepare the necessary foundation by drills.

3. The teachers offer several suggestions as to improving the "Pedagogy" involved in teaching fractions.

a. Omit the the use of the "pen" in teaching addition and subtraction of fractions.

b. Require absolute accuracy in every particular in all mathematical work.

c. Give a good share of time to the little details in mathematics.

d. If more of the problems involving the solution of an equation in algebra, especially the drill exercises, had fractional results the skill to work with fractions would be increased due to the experience gained in checking the equations. Too many problems in most texts are made to order and come out even.

e. If in all work with the fundamental operations, a product is called a product; a dividend is called a dividend; a divisor is called a divisor, etc., the problem would have greater meaning, and there would be fewer mistakes in number combinations. Often the pupil knows he must invert the divisor, but does not know what part of the problem is divisor and what part is dividend. If asked he could tell you the rule, but it has no meaning when he tries to apply it to a problem.

f. Eliminate the word cancel from the vocabulary of the student, also the words "cast out like factors," etc. There are only four fundamental operations, why not say divide both numerator and denominator by a common factor, then the pupil knows what he is doing if he understands division.

g. In teaching the fundamental operations with fractions, graphs are an effective agent in giving meaning to the work. If the student is made to understand the real meaning of $\frac{3}{4}$ of

$\frac{5}{7}$, or $\frac{3}{4} \times \frac{4}{7}$ as contrasted with $\frac{3}{4}$ plus $\frac{4}{7}$, he is much more apt to have clear sailing. This can be accomplished best by the use of graphs.

Conclusion

At the beginning of the study we set before ourselves several problems.

1. Is it true that the high school student does not know the simple processes in fractions? As the median grade in the 9th grade is 14.7, we are safe in assuming that he does know the simple processes. His trouble begins when the examples begin to be complex.

2. Is he capable of solving examples in fractions, required as a foundation for algebra? This foundation should include ability to solve most of the problems in the test. This he is unable to do, and, therefore, it is necessary for the algebra instructor to "review," systematically and make up the handicap.

3. As far as the six-year high school is concerned, where is improvement made in fraction-solving ability? In the 7th and 8th grades. Here there is marked improvement. After the 8th grade is passed there is little progress.

Problems 4 and 5 have to do with types of errors and methods of improvement, and these have been discussed in the body of the paper.

NEWS AND NOTES

The following preliminary report was submitted by the Committee on Examinations of the Association of Mathematics Teachers of New Jersey, at the regular spring meeting of the association.

Part I. Curricula Content.

Grades I-VI. The committee recommends a content for these grades which shall be uniform and constitute a basis for a proper course in grades VII-IX. This basis cannot neglect a proper allowance of time for work in fractions and in problem solving.

Grades VII-IX. The content for these grades cannot meet its proper functions if a large part of it must be given over to remedying the deficiencies of the preparation in grades I-VI. The committee recommends that the proper function for these grades will be met by the adoption of the recommendation of the National Committee on Mathematical Requirements with the exception of the recommendation of demonstrational geometry of the usual type. (The association has already voted to make this exception.) *This recommendation covers the 8-4 type of school as well as the 6-3-3.* It is desirable that one of the five types of organization given by the National Committee be adopted. Plan A, with the deletion of demonstrational geometry and the inclusion in its stead of a unit of arithmetic dealing with relations expressible by fractions, decimals and per cent, appears to be the best type of organization.

Grade X. Beyond any question the content of this grade should consist of a year of algebra. As most schools can offer but one content it is probable that the definition of Mathematics A of the College Entrance Examination Board must be adopted. The work of his year should be elective for high school credit.

Grade XI. The content in this grade should consist of a year of demonstrative geometry. The committee does not see any great advantage to be gained by the amalgamation of plane and solid geometry into a single year, but rather much possible harm. In the best interest of all concerned, the practice of skimming through important segments of mathematics should be

abandoned. The aim in this grade is a thorough-going scholarship. The proper place for an extensive, rather than an intensive course of study is in grades VII-IX. It will be found that the added year of maturity of the student will make possible a much more satisfactory year of geometry than now commonly obtains. The work of this year should be elective.

Grade XII. The content in this grade is unsatisfactory and will likely continue to be so owing to the conflicting college entrance interests of the available students. Under the program herein recommended for grades I-XI, courses in solid geometry, plane trigonometry, advanced algebra, and review mathematics will be offered in grade XII as the need arise. The committee, because of its view that grades X-XII should develop a sound scholarship, cannot regard with favor the proposals of the National Committee to distribute the subjects mentioned above in order to permit the adoption of a unit of calculus in grade XII. In time, perhaps, it will be possible to offer a year's course in elementary functions. In this connection the committee desires to point out that one obstacle to such an adoption consists in college entrance credits.

Part II. Extra Mural Examinations.

I. The committee favors as a general principle the maintenance of examination systems.

II. Examinations in New Jersey.
The committee recommends:
1. A state-wide examination at the end of grade VI.
2. A state-wide examination at the end of grade IX.
3. All such examinations should take the form of composite power tests of important component abilities.
4. The present state-wide examination in arithmetic for the "highest elementary grade" (grade VIII) does not serve any useful purpose in form or results and should be discontinued.

III. College Entrance Examinations.
The committee finds that
1. College entrance, apart from considerations of moral character and physical health, should depend upon:
a. General intelligence.

b. Secondary training.
c. Specific preparation for advanced study.

2. Tests of general intelligence are available.
3. General secondary training is not suitable for examination, for it is dependent, at least in part, upon:
   a. General habits of the candidate.
   b. The course of study pursued.
   c. The candidate's standing in the school.
   d. The character of the school itself.
4. Specific preparation for advanced study is a proper subject for examination. In this connection the committee notes:
   A. The present examinations do not give the largest possible measure of specific preparation for
      a. The form of the examination is in fault as a measuring instrument. (See Wood, "Reliability and Difficulty of C. E. E. B. Examinations in Algebra and Geometry," 1921, published by the Board.)
      b. The topics chosen are not weighted in accordance with their value as a preparation for advanced study. (See Summary of the Report of National Committee, Bulletin No. 32 of 1921, Bureau of Education, Dept. of Interior, Washington. p. 38.)
      c. The present "comprehensive" examinations are not strictly comprehensive, that is, they are not composites of tests which individually test important topics or abilities.
   B. The proper form of examination, and the scoring of the same, is considered in part III of this report.

Part III. Principles of Examinations and Grading.

I. The aim of all testing is to show correct differences or ratios.
   A. This aim is not met by the "long-written" examination for the sampling is too slight. See
      1. Wood, same reference as above.
      2. McCall, "How to Measure in Education," part II.
      3. Thorndike, "Mental and Social Measurements," Chap. II.
   B. This aim cannot be accomplished if undistributed scores arise.

1. Two zero scores are undistributed.
2. Two perfect scores are undistributed.
3. Effect of:
   a. "Right-wrong" scoring.
   b. Coarse scoring.
   c. Time allowance.

C. The composite test increases the accuracy of the sampling.
1. Consider the extent to which any good intelligence test is a composite.
2. Consider the correlations obtained by such tests with sound criteria. What a test measures is determined by its correlations.
3. Principles of construction.
   a. All important component fields tested.
   b. Number of responses multiplied.
   c. Scoring objective.
   d. The form must be prepared in advance. This permits of the incorporation of the results of the previous use.

II. Finding the composite score from the scores of the several component tests.

A. It is not the absolute size of the scores but their fluctuations which causes the composite to vary. See McCall, p. 30.

B. A convenient measure to use of the extent to which any set of scores will affect the composite is half the variation of the middle 50% of scores. This quantity is known as "Q" or the "semi-quartile range." Formula:

$$Q = \frac{Q_3 - Q_1}{2}$$

If Q for any set of scores is twice as large as it is for another set of scores the former will have twice the effect of the latter upon the composite.

C. Assign the desired weight to each set of component scores and then add to get the composite score.

III. Grading the composite scores.

A. General principles of measuring.
1. Measuring in the mental field does not differ in kind from measuring in the physical field but only in complexity.

2. From a definite reference point.
   a. Absolute zero.
   b. An arbitrary point.
   c. The mean of the scores.
3. With a defined unit of the same kind of thing.
4. By relative position.
   a. Percentiles.
   b. Quartiles.
   c. Median.

B. Present methods.
1. Reference points.
   a. Perfection. "100%."
   b. Zero. (Neither obsolute nor defined.)
   c. Passing grade. (A percent not a percentile.)
2. Unit. None. A percent of an examination is not a unit.
3. Adanvtages.
   a. Scoring is simple.
   b. A numerical score. (Except the A, B, C, D, E system.
   c. In general use.
4. Disadvantages.
   a. No actual comparability. (Save by chance.)
   b. Unreliable.
   c. Unjust.

C. McCall's T scale method. (Cited as an illustration.)
1. Advantages.
   a. Comparability. Scores may be combined by simple addition.
   b. Based on difficulties surmounted.
2. Method of computation. McCall, pp. 290-291.
   a. Find the composite score for each pupil.
   b. Compute the percent of pupils *exceeding plus one-half those reaching* each composite score.
   c. Convert these percents into T scores by means of a table given on page 274.
   d. Give each pupil his scale score instead of his composite score.

D. The A, B, C, D, E system.
1. Disadvantages.
   a. Not a numerical system. Statistical methods do not apply. What does 1A, 2B, 1C, 1D, 3E equal? The

answer cannot be found unless the value of each letter at its midpoint is known, that is, unless the system is made numerical.

b. Too few groups.

i. In case of an error in grading of but one step the pupil is displaced relatively ⅕ of the entire range.

ii. Too few groups to figure correlations or make other useful comparisons. "What a test measures is determined by its correlations."

2. Improvements.

a. Adopt more steps. Thirteen are necessary if correlations are not to be adversely affected by coarse grading.

b. The midpoint of the middle group (mid C in the A, B, C, D, E system) should indicate the average performance of a sufficiently large group.

c. The steps in the scale should be equal. A — B = B — C = etc. This requires that the definition of the steps be referred to some function of the variability of the scores.

IV. The reliability of a measure of central tendency is due to the number of scores as well as to their variability.

A. The number of scores can be increased when the variability cannot be decreased. *Measure the whole group, not a class.*

V. Questions for Members of the Association:

A. Would you favor an appeal to the State University for extension courses in the theory of mental measurements?

B. Would you enroll in such courses?

C. Would you attempt to enlist the support of your colleagues for such courses?

Respectfully submitted,

COMMITTEE ON EXAMINATIONS.

P. W. Averill,
Josephine B. Emerson,
Howard F. Hart,
Chairman,
Carl N. Schuster,
Harrison E. Webb.

Reading Course for Secondary School Teachers of Mathematics

At the annual meeting of the Inland Empire Council of Teachers of Mathematics, held at Spokane in April, a report was given by a committee which had been appointed to suggest a desirable course of reading for teachers of mathematics in the secondary schools of the four northwest states (Oregon, Washington, Idaho, Montana) who are ambitious to advance themselves professionally during the year.

The committee gave five brief lists of groups of books which they considered suitable for this purpose, suggesting either that one book from each group be read during the ensuing year, or else that emphasis for the year be placed on the books of a single group. After some discussion the Council voted to recommend the philosophy group for reading during the year and as a basis for a portion of next year's program.

The list of books suggested by the committee was as follows:

Algebra and Geometry—Elements of Non-Euclidean Geometry, Sommerville, Open Court; Non-Euclidean Geometry and Trigonometry, Carslaw, Longmans; Projective Geometry, Dowling, McGraw-Hill; Synthetic Projective Geometry, Lehmer, Ginn & Co.; Number System of Algebra, Fine, D. C. Heath Co.; Non-Euclidean Geometry, Bonola, Open Court.

History—History of Mathematics, Cajori, Macmillan; History of Mathematics, Ball, Macmillan; History of Elementary Mathematics, Cajori, Macmillan.

Pedagogy—Teaching of Elementary Mathematics, Smith, Macmillan; Teaching of Mathematics, J. W. A. Young, Longmans; Teaching of High School Mathematics, George W. Evans, Houghton-Mifflin; The Teaching of Geometry, Smith, Ginn & Co.

Philosophy—Mathematical Philosophy, Keyser, Dutton; Human Worth of Rigorous Thinking, Keyser, Columbia U. Press; Mysticism and Logic, Bertrand Russell, Longmans; Introduction to Mathematical Philosophy, Bertrand Russell, Macmillan; Mysticism in Modern Mathematics, Berkeley, Oxford U. Press; The Foundations of Science, Poincaire, The Science Press.

Miscellaneous—Creative Chemistry, Slosson, Century; The Friendly Stars, Martin, Harper; The Mind in the Making, J. H. Robinson, Harper; The Place of Science in Modern Civilization, Veblen, Huebsch; Monographs on Modern Mathematics, J. W. A.

Young, Longmans; Mathematical Recreations, Ball, Macmillan; Scrap Book of Mathematics, White, Open Court; Memorabilia Mathematics, Moritz, Macmillan.

(Signed) Professor C. A. Isaacs, Pullman, Wash., Chairman; C. W. Vanderwalker, State Normal, Lewiston, Idaho; Ella M. Spaford, High School, Butte, Mont.; Clarice Van Eman, Fairhaven High School, Bellingham, Wash.; R. M. Winger, University of Washington.

NEW BOOKS

High School Algebra. By C. E. Rushmer and C. J. Dence. American Book Company, New York City, 1923. Pp. 395.

The new courses in ninth grade mathematics are placing less emphasis upon the mere manipulative side of algebra; they introduce a more humanized, vital body of subject matter which offers a greater challenge to the interest and effort of the pupil. This new text in elementary algebra approaches the ideas that the National Committee has set up except in regard to the omission of (1) the introduction of numerical trigonometry, and (2) a more conscious attempt to teach the concept of *relationship*. The authors call attention to these characteristics of the text:

1. Simplified subject matter.
2. Orderly sequence of topics.
3. Emphasis of the close relation between arithmetic and algebraic processes.
4. Postponement of the idea of negative number until a thorough foundation in the four fundamental processes is established.
5. Abundance of drill exercises, especially in processes in which students experience difficulty.
6. Many exercises in alebraic representation as an aid to the solution of problems.
7. Problems which have some human interest or illustrate some useful application of physics or geometry.
8. Graphic methods of representing data and solving problems.
9. Increased use of the formula.
10 Frequent use of literal exponents as an aid in fixing the laws for exponents.
11. Frequent use of decimals.
12. Comprehensive reviews.

College Board and Regents, Questions and Answers, in Geometry. College Entrance Publishing Company, 344 West 38th Street.

This book contains thirty-six examination papers in Plane Geometry given by the College Entrance Examination Board and by the Board of Regents, of the New York State Department of Education since 1910. Twenty-five of the papers contain diagrams and answers. The answers were prepared under the supervision of an editorial board of high school teachers, some of whom are readers for the College Entrance Examination Board and for the Board of Regents. The teacher will find that the answers have been so framed as not to lend themselves to memorizing.

The book contains the complete Geometry Syllabus (based on the Harvard University Syllabus) prepared by the New York State Department of Education and now in force, and also a summary of methods for solving "originals."

An Introduction to Projective Geometry. By R. M. Winger, Ph.D. D. C. Heath & Co. (1923). xiii + 425 pp. $4.40.

Winger's Projective Geometry embodies a year's course of lectures for university students—seniors or graduates. It is also intended, says the author, "for the prospective teacher who is seeking a proper orientation of elementary mathematics." The first six chapters, dealing with projective geometry as the main topic, depend less upon geometrical intuition than upon a lucid analysis, and can be read without extensive mathematical equipment; but the exercises are at times rather stern. This topic leads in the succeeding chapters to modern higher algebra, and in the last chapter to non-Euclidean Geometry. There are occasional references to more extensive treatises, and to mathematical literature too recent to be in the books. Those of us who remember when college mathematics was divided into well-fenced fields can welcome an author who reaches anywhere for a method he finds useful—and who, if this method may appear strange, either explains it then and there, or refers to an accessible explanation. The "prospective teacher" may well spend his arduous leisure on this concise, clear, and interesting book.

CHARLES W. EVANS.

"We have always understood that mathematics is the most exact of all sciences and its theorems capable of absolute proof, and yet here in the gravitational field we find a space in which the proven theorems of our school geometry do not hold good any more." This statement by Dr. Steinmetz[1] is followed by the explanation, viz., that mathematics is an exact science and its conclusions are capable of absolute proof simply because all mathematical conclusions are relative, being conditioned by certain premises or "axioms." To the mathematics teacher, this is likely to be familiar ground if he be conversant with the work of Gauss, Lobatschewsky and others, who nearly a century ago had shown that Euclid's geometry is not the only

[1] Charles P. Steinmetz, "Four Lectures on Relativity and Space." Mc-

possible one, that at least two other different sets of axioms, leading to a different non-Euclidean geometry (the elliptic and hyperbolic) fully as consistent as Euclid's, could be formulated.

However, to most mathematics teachers, this geometry has been wholly speculative; hence it is in this connection that this book is of significance to them. It will be recalled that speculation was rife as to which of the three geometries completely represents the space of physical nature. But, living in our physical space which, as far as our sense perceptions extend, is Euclidean or zero-space, we are apt to forget that we have adopted the Euclidean interpretation not on the basis of truth but rather on that of convenience, since we have not been able hitherto to prove by experiment that these other interpretations are truer or more exact than the one now in use. To the mathematics teacher to whom this statement is repugnant or that of Poincaré (Science and Hypothesis, p. 65)—"Our mind had adopted the geometry most advantageous to the species, or, in other words, most convenient. Geometry is not true, it is advantageous"—this book will be a tonic.

The answer to the question raised above has now been given indirectly by the relativity theory showing that physical space varies between the Euclidean and the *elliptic*. The accumulated data, notably during the last solar eclipse, tend to confirm this theory as developed by Einstein. Thus the non-Euclidean geometry has at last become basic to a wider conception of physical space.

In this simple yet authoritative discussion, the author gives by analogy, example and comparison a general conception of the theory and its fascinating deductions and conclusions. Thus he has succeeded admirably in presenting, in relatively simple mathematics, a theory which is intrinsically mathematical and too abstruse otherwise.

This book should be read by every teacher of mathematics and science who is interested in this enlarging conception of the physical universe and in the basic role played therein by a hitherto purely speculative field of geometry.

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